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Professor A. S. EDDINGTON, F.R.S., PRESIDENT, in the Chair.

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Thomas George Hirst, High Barn, Birstal, near Leicester;
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William John Thorrowgood, M.I.E.E., M.I.R.S.E., 2 Waldemar Road, Wimbledon Park, London, S.W. 19;
Harihar Parames Waran, M.A., Christ's College, Cambridge; and
Dr. Jan Woltjer, Conservator at the Observatory of Leiden, Holland,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Edward Neville da Costa Andrade, D.Sc., Ph.D., F.Inst.P., Professor of Physics, Artillery College, 18 Keyes Road, Cricklewood, London, N.W. 2 (proposed by H. C. Plummer);

Weld Arnold, formerly Fellow in Astronomy, Harvard University, 2A Alexander Square, London, S.W. 3 (proposed by A. R. Hinks);

P. Carrasco, Observatorio Astronómico, Madrid, Spain (proposed by C. P. Butler);

Miss Caroline Carter, B.A., Brooklyn, Lion Road, Bexley Heath, Kent (proposed by P. J. Melotte);

George Stanley Le Beau, M.A., Mathematical Department, East London College, University of London, and 27 Rusthall Avenue, Bedford Park, London, W. 4 (proposed by A. Fowler);

Alexander John Thompson, B.Sc., Assistant Statistical Officer, General Register, Somerset House, London, W.C. 2 (proposed by T. C. Hudson);

A. Vela, Observatorio Astronómico, Madrid, Spain (proposed by C. P. Butler); and

Mrs. Lilian R. Widgery, B.A., 102 Elliscombe Road, Charlton, London, S.E. 7 (proposed by P. J. Melotte).

Fifty-eight presents were announced as having been received since the last meeting.

The Motions of the Stars in a Kapteyn-Universe.

By J. H. Jeans.

1. In his recent paper "On the Distribution of the Stars in Space,"* and again in his "First Attempt at a Theory of the Arrangement and Motion of the Sidereal System,"† Professor Kapteyn has given a masterly summary of our present knowledge, mainly due to his own researches, of the arrangement of the stars in space. In brief, he finds that the arrangement of the stars may be regarded, with very tolerable accuracy, as consisting of a system of spheroidal shells, each of constant density. The spheroids are similar and similarly situated; they are oblate, having their axes in the ratio 5·102 : 5·102 : 1, their equal longer axes being in the galactic plane and their common centre being not far removed from the position of the sun.

In the second of these papers Professor Kapteyn attempts a dynamical investigation of such a system of stars. He assumes that the stars are in a dynamically steady state, and makes the further supposition that along the shortest axes of these spheroids (*i.e.* in the directions of the galactic poles) the law of distribution is Maxwell's well-known law for the distribution of molecular velocities of a gas in isothermal equilibrium.

Kapteyn's first assumption can neither be proved nor disproved; in the present state of our knowledge it is probably the most reasonable assumption to take as the basis of a dynamical discussion. But, for

* *Astrophys. Journ.*, 52 (1920), 23. † *Ibid.* (1922).

reasons which will appear later, the second assumption seems much less plausible. Indeed, in introducing it Kapteyn writes: "We will now introduce some considerations from the kinetic theory of gases, the applicability of which to the stellar system might be considered doubtful. . . . The results which will be derived cannot lay claim to be demonstrably correct."

The main object of the present paper is to show that the introduction of Kapteyn's dubious second assumption is unnecessary. As we shall now see, the principal features of the motion of the stars can be deduced from a knowledge of their distribution in space, combined only with the single assumption that the system of stars is in a steady state.

2. Kapteyn's given distribution is one of symmetry about a plane (the Milky Way) and about an axis (the pole of the Milky Way). Taking the centre of the system as origin and the plane of the Milky Way as the plane $z = 0$, we introduce cylindrical co-ordinates ϖ, θ, z, and denote the velocities of a star in the directions of ϖ, θ, z increasing by Π, Θ, Z.

Let the stars be of various distinct types, distinguished by the suffixes 1, 2, 3. . . .

Let the law of distribution of stars of type 1 at any instant t be assumed to be

$$f_1(\Pi, \Theta, Z, \varpi, \theta, z, t)d\Pi\, d\Theta\, dZ\, d\varpi\, \varpi\, d\theta\, dz \quad . \quad . \quad (1)$$

this representing the number of stars in the element of volume $d\varpi\, \varpi\, d\theta\, dz$ of which the velocity components at time t lie within the small range between Π and $\Pi + d\Pi$, Θ and $\Theta + d\Theta$, Z and $Z + dZ$.

Each star will be in motion under gravitational forces which may be regarded as made up of two parts—

(A) The forces from adjacent stars.

(B) The forces from the gravitational field of the universe as a whole.

A simple calculation shows that in our own system of stars forces (A) are in general quite insignificant in comparison with forces (B).* Thus we may expect a system of stars such as our own to tend first to assume a state of steady motion under forces (B). When such a state has been reached the average effect of forces (B) becomes zero, and forces (A) now come into play, until gradually the system assumes a final steady state under both forces (A) and (B). This final steady state is exactly analogous to Maxwell's steady state in the kinetic theory of gases; under it the velocities of the stars in any small region of space conform to Maxwell's law of distribution of velocities, so that these can show no preference for special directions in space, and there can be no star-streaming. The fact that star-streaming is observed in our universe affords conclusive proof that this final state has not yet been attained, a conclusion which is fully in accordance with calculations as to the time which would be required for its attainment.†

On the other hand, the time required for the attainment of a steady state under forces (B) alone is of an altogether lower order of magnitude, so that it is quite possible that a steady state of this kind has already

* *M.N.*, 74 (1913), 109. † *L.c. ante*, p. 112.

been attained. In agreement with Kapteyn, we shall assume this to be the case throughout the remainder of the present paper.

3. For such a steady state it is readily shown * that the law of distribution (1) assumes the special form

$$f_1[\tfrac{1}{2}(\Pi^2+\Theta^2+Z^2)-V,\varpi\Theta]d\Pi\,d\Theta\,dZ\,d\varpi\,\varpi\,d\theta\,dz \quad . \quad (2)$$

where V denotes the gravitational potential of the universe as a whole.

In this formula, f_1 is no longer specified as depending explicitly upon t, but it will be understood that the action of the feeble forces (A) results in a slow secular change in the constants which occur in the specification of f_1. It is easily shown that this formula demands star-streaming in the direction of Θ, the direction which Charlier and Kapteyn assign as the axis of star-streaming in our system.

Let ν_1 denote the number per unit volume of stars of type 1 in any small region of space, and let $(\Pi\Theta)_1$ denote the average value of the product $\Pi\Theta$ for these stars. Then, under the law of distribution (2),

$$\nu_1(\Pi\Theta)_1=\iiint_{-\infty}^{+\infty}f_1\Pi\Theta\,d\Pi\,d\Theta\,dZ=0,$$

and similarly

$$\nu_1(\Theta Z)_1=0,\quad \nu_1(Z\Pi)_1=0,\quad \nu_1(\Pi^2)_1=\nu_1(Z^2)_1.$$

Let us introduce quantities p_1, q_1 defined by

$$\nu_1(\Pi^2)_1=\nu(Z^2)_1=p_1,\quad \nu_1(\Theta^2)_1=q_1 \quad . \quad . \quad . \quad (3)$$

Taking u, v, w for the moment to represent components of stellar velocity along any three rectangular axes x, y, z, it is easy to deduce from the law of distribution (2) three equations of the type †

$$\frac{\partial}{\partial x}[\nu_1(u^2)_1]+\frac{\partial}{\partial y}[\nu_1(uv)_1]+\frac{\partial}{\partial z}[\nu_1(uw)_1]=\nu_1\frac{\partial V}{\partial x} \quad . \quad . \quad (4)$$

these being, in point of fact, the hydrodynamical equations of motion for the ν_1 stars of the first type in a unit volume at x, y, z, taking into account the fact that forces (A) are negligible.

Transforming to the cylindrical co-ordinates ϖ, θ, z already employed, these equations become

$$\frac{\partial p_1}{\partial z}=\nu_1\frac{\partial V}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{\partial q_1}{\partial \theta}=\nu_1\frac{\partial V}{\partial \theta} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{\partial}{\partial \varpi}(\varpi p_1)-q_1=\varpi\nu_1\frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad (7)$$

Since the field of stars is assumed to be symmetrical about the axis of z, $\partial V/\partial\theta$ must vanish, and equation (6) merely expresses the obvious

* *Problems of Cosmogony and Stellar Dynamics*, p. 236.
† *Ibid.*, p. 232.

fact that q_1 cannot vary with θ. It is convenient to rewrite equation (7) in the form

$$\frac{\partial p_1}{\partial \varpi} + \frac{p_1 - q_1}{\varpi} = \nu_1 \frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and equations (5) and (8) now contain the material for discussion.

The equations, as at present written, refer only to stars of one special type, but we are free to add together corresponding sides of any number of such equations. In this way we obtain equations such as

$$\frac{\partial p}{\partial z} = \nu \frac{\partial V}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{\partial p}{\partial \varpi} + \frac{p - q}{\varpi} = \nu \frac{\partial V}{\partial \varpi} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where p, q, ν now refer to all stars of any type or class desired, *e.g.* all B-type stars, or all luminous stars or all dark stars. But V is in every case the gravitational potential of all stars and of all dark matter in the universe. It will be noticed that equations (9) and (10) are no longer hydrodynamical-equations.

Kapteyn's investigation gives us in effect the values of ν at every point of the universe. If we assume the stars to have everywhere the same known average mass, it is a mere matter of calculation to determine V at every point. This being done, equation (9) enables us to find p at each point, and equation (10) then determines q at each point. Since ν is already known, a knowledge of p and q gives the mean values of Π^2, Z^2, and Θ^2 at each point of space and for any special class of stars we please. The mean values of Π^2 and Z^2 will give us information as to the mean peculiar motions of the stars, while the mean value of $\Theta^2 - Z^2$ will give information as to the extent of star-streaming.

4. As a preliminary illustration of the use of our formulæ let us consider their application to the B stars. Charlier * has shown that these are confined to a flat, biscuit-shaped space surrounding the galactic plane. The sun is only about 90 parsecs from the centre of this region, its radius in the plane of the galaxy being about 500 parsecs and its extension on either side of this plane only about 150 parsecs. As a matter of convenience let us suppose the boundary of this region to coincide with the first ellipsoidal shell of Kapteyn, of which the boundary is

$$\frac{\varpi^2}{a^2} + \frac{z^2}{c^2} = 1, \qquad \text{where} \begin{cases} a = 602 \text{ parsecs} \\ c = 118 \text{ parsecs} \end{cases} \quad . \quad . \quad (11)$$

Let us further, as an approximation which appears to be in very fair agreement with the facts, assume the B-type stars to be arranged in similar ellipsoidal shells inside this one, so that the density ν of B-type stars is of the form

$$\nu = \phi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad (12)$$

* *Lund Meddel.*, 2 (1916), No. 14.

Inside the ellipsoid occupied by the B-type stars the potential of the whole universe, as arranged by Kapteyn, is of the form

$$V = C - A\varpi^2 - Bz^2,$$

where the last two terms arise from the potential of the ellipsoid itself and all external ellipsoids add only a constant term to V.

From equation (9) we now have

$$p = \int \nu \frac{\partial V}{\partial z} dz = -\int \phi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right) 2Bz\, dz \quad . \quad . \quad (13)$$

If we put $\int \phi(x)\, dx = \psi(x)$ and adjust the additive constant so that $\psi(x)$ vanishes when $x = 0$, then equation (13) becomes

$$p = Bc^2 \psi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad (14)$$

no function of ϖ being added, since p must vanish at the boundary (11). Equation (10) now gives

$$\frac{q - p}{\varpi} = \frac{2\varpi}{a^2}(Aa^2 - Bc^2)\phi\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right)$$

or

$$\frac{q - p}{\nu} = 2(Aa^2 - Bb^2)\frac{\varpi^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The maximum value of p occurs at the centre of the system, being $Bc^2\psi(1)$ or $Bc^2\int_0^1 \phi(x)\, dx$. If we assume, for illustration, a law of density

$$\nu = \nu_0\left(1 - \frac{\varpi^2}{a^2} - \frac{z^2}{c^2}\right),$$

then $\phi(x) \propto \nu_0(x)$ and $\psi(1) = \frac{1}{2}\nu_0$. In this case the value of p/ν at the centre is $\frac{1}{2}Bc^2$. Had we assumed uniform density we should have had $p/\nu = Bc^2$ at the centre.

In his numerical calculations Kapteyn takes the parsec as unit of length and the gravitational mass of the "average star" as unit of mass. In these units he calculates the gravitational forces at the points $(a, 0)$ and $(0, c)$, and finds

$$2Aa = 33{\cdot}19, \qquad 2Bc = 40{\cdot}06.$$

In these units the value of p/ν at the origin is

$$p/\nu = \tfrac{1}{2}Bc^2 = 1182.$$

To reduce to ordinary C.G.S. units we have to multiply by $\gamma M/l$. Here l is the number of cms. in a parsec ($3{\cdot}08 \times 10^{18}$), γ is the gravitation constant ($6{\cdot}66 \times 10^{-8}$), and M is the mass of the "average star" in grammes. More exactly this last quantity should be the mass of the average visible star increased by the ratio of dark matter to visible stars

in the universe. Later on we shall determine the value of M as $6{\cdot}34 \times 10^{33}$ grammes. We shall use this value here, so that

$$\frac{\gamma M}{l} = 1{\cdot}37 \times 10^{8},$$

and

$$\frac{p}{\nu} = \tfrac{1}{2}Bc^2 = 16{\cdot}2 \times 10^{10} \text{ (cms. per sec.)}^2.$$

This quantity is of course equal to the mean value of Z^2 or Π^2 for all the B-type stars near the origin. We see that even close to the origin, where they attain their maximum values, Z and Π are only of the order of 4 kms. a second (or 6 kms. a second in the extreme case of uniform distribution). It is at once clear that the low velocities of the B-type stars are fully accounted for by their observed distribution in space: it is not necessary to interpret them as a consequence of their supposed exceptional mass.

Independently of the law of distribution of density, equation (15) gives

$$\frac{q-p}{\nu} = 15{,}252\left(\frac{\varpi}{a}\right)^2 = 209 \times 10^{10}\left(\frac{\varpi}{a}\right)^2 \text{ (cms. per sec.)}^2.$$

This quantity represents the mean value of $\Theta^2 - Z^2$ or $\Theta^2 - \Pi^2$. When it has a positive value it is clear that there must be star-streaming in the directions of θ increasing and decreasing. If U′, U″ are the velocities of the two star streams, we shall see later (§ 6) that $\frac{q-p}{\nu}$ is equal to U′ U″. Thus in the special case now under discussion the velocities of the star streams are of the order of 14·5 kms. a second at the boundary $\varpi = a$ (602 parsecs) and decrease uniformly with ϖ, precisely as though the two star streams were rigid bodies rotating in opposite directions. At Charlier's position for the sun ($\varpi = 90$ parsecs), the velocity of star-streaming for the B-type stars will be only about 2·2 kms. a second.

5. Let us now apply the same method to the general mass of the stars. From equation (9) we have

$$p = -\int \nu \frac{\partial V}{\partial z} dz \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where the integration is along a line parallel to the axis of z until a region is reached at which the star density ν is negligible. The integral can only be evaluated by quadratures. Kapteyn's investigation gives the value of ν at every point, and he has calculated the value of $-\partial V/\partial z$, which he denotes by $G \cos \phi$, at certain selected points. I have first evaluated p at points along the galactic axis ($\varpi = 0$), taking the thicknesses of Kapteyn's various "shells" as the elements of integration in equation (16). The calculation is exhibited in Table I. The entries Ex. 1, Ex. 2, etc., refer to shells obtained by extrapolation beyond the ten shells of Kapteyn. No estimate is made of the density or thickness of these shells, the extrapolation being only with reference to their contribution to p. The last column gives the square root of the mean value

of Z^2 or Π^2 in kilometres per second on the assumption that M has the value already assigned to it in § 4. This may be regarded as giving approximately one component, other than Θ, of the star's peculiar velocity; it is seen to range from about 13 kms. a second in the close neighbourhood of the origin, through a value of 17 kms. a second at about 300 parsecs from the origin, to a final value of about 23 kms. a second, which is approximately attained at about 1600 parsecs from the origin.

TABLE I.

Values of p and Z along Galactic Pole.

Shell.	Mean ν.	Mean $\frac{\partial V}{\partial z}$.	dz in parsecs.	$\nu\frac{\partial V}{\partial z}dz$.	Ellipsoid.	z in parsecs.	p.	ν.	$\frac{p}{\nu}=(Z^2)$.	Z (kms. per sec.).
1	·036	22·0	118	93·5	Centre	0	504·0	·0451	11,200	12·4
2	·023	48·0	80	88·3	I.	118	410·5	·0285	14,400	14·0
3	·014	58·5	98	80·3	II.	198	322·2	·0180	17,900	15·7
4	·0090	64·8	117	68·2	III.	296	241·9	·0113	21,400	17·1
5	·0057	66·3	140	52·9	IV.	413	173·7	·00715	24,300	18·2
6	·0036	65·0	164	38·4	V.	553	120·8	·00451	26,800	19·2
7	·0023	62·3	185	26·5	VI.	717	82·4	·00285	29,000	19·9
8	·0014	58·5	212	17·4	VII.	902	55·9	·00180	31,100	20·7
9	·0009	54·0	251	12·2	VIII.	1114	38·5	·00113	34,100	21·6
10	·0006	48·7	295	8·6	IX.	1365	26·3	·000715	36,800	22·4
Ex. 1				6·1	X.	1660	17·7	·000451	39,300	23·2
Ex. 2				4·2	Ex. I.		11·6	·000285	40,700	23·6
Ex. 3				2·9	Ex. II.		7·4	·000180	41,100	23·7
					Ex. III.		4·5	·000113	40,000	23·4

The next stage in the computations was to construct tables giving the values of p throughout the whole system. The method adopted was to regard p at any point, as before, as arising from contributions from all the shells standing outside this point. It was found that the contribution $\nu\frac{\partial V}{dz}dz$ from any specified shell did not vary greatly for different points of this shell until very low galactic latitudes were reached. For instance, for shell 2, if the contribution in latitude 90 is taken to be 1·000, those in latitudes 75°, 60°, 45°, 30°, 15°, and 0° are respectively ·999, ·996, ·990, ·97, ·90, and ·67; for shell 8 the corresponding numbers are ·984, ·950, ·86, ·745, ·57, and ·27. From a table giving the ratio of contribution in any latitude to that in latitude 90° it was found easy to evaluate p at any desired point. The following table gives the values found for p and p/ν at a series of points in the galactic plane, the points selected being the boundaries of the various shells. The table also contains the value of Z, the square root of (Z^2) in kilometres per second, the value of $(q-p)/\nu$ calculated from equation (10), and the square root of (Θ^2-Z^2) in kilometres per second.

TABLE II.

Values of p, Z, q, and Θ in Galactic Plane.

Ellipsoid.	ϖ in parsecs.	ν.	p.	$\frac{p}{\nu}$.	$\frac{q-p}{\nu}$.	Z (kms. per sec.).	$\sqrt{(\Theta^2-Z^2)}$ (kms. per sec.).
Centre	0	·0451	504	11,200	0	12·4	0
I.	602	·0285	368	12,900	14,300	13·3	14·0
II.	1010	·0180	251	13,900	25,900	13·8	18·8
III.	1510	·0113	165	14,600	38,500	14·1	23·0
IV.	2106	·00715	102	14,400	51,500	14·0	26·6
V.	2820	·0045	63	14,000	62,300	13·8	29·2
VI.	3656	·00285	36	12,600	69,100	13·1	30·8
VII.	4600	·00180	21	11,600	78,200	12·6	32·7
VIII.	5675	·00113	12	10,600	83,400	12·1	33·8
IX.	6960	·000715	7	9,900	86,300	11·7	34·4
X.	8465	·00045	4	8,900	86,300	11·0	34·4

The tables have given the values of p, Z, q, Θ only in the galactic plane and along the galactic axis. It is easy to obtain a rough idea of the values of these quantities in other parts of space. As regards p, we have already noticed that the contribution of any individual shell to p remains approximately constant until low galactic latitudes are reached, so that we may think of p as remaining constant over any shell except in low galactic latitudes. In other words, except in these low latitudes p and Z may be regarded as functions of ν only, and the last four columns of Table I. may be regarded as tabulating p and Z in terms of ν. In low galactic latitudes Table II. may, of course, be used in a similar manner. As regards q, we obtain from equations (9) and (10) on elimination of V

$$\frac{\partial}{\partial z}\left(\frac{q-p}{\nu\varpi}\right)=\frac{\partial p}{\partial \varpi}\frac{\partial}{\partial z}\left(\frac{1}{\nu}\right)-\frac{\partial p}{\partial z}\frac{\partial}{\partial \varpi}\left(\frac{1}{\nu}\right).$$

We have seen that in fairly high galactic latitudes p may be regarded as a function of ν, in which case the right hand of this equation vanishes. Again, in low latitudes $\partial p/dz$ and $\partial \nu/dz$ are very small, so that again the right-hand member is small. We may get a rough idea of the extent of star-streaming by neglecting the right-hand member of this equation altogether, in which case we find that $(q-p)/\nu$ becomes a function of ϖ only, and Table II. may be regarded as giving this quantity in terms of ϖ.

6. If we regard the stars in any small region of space as forming two star drifts in the direction of θ increasing and decreasing, if ν', ν'' are the densities of these streams, U′, − U″ the stream velocities, and if u', v', w' and u'', v'', w'' are the peculiar velocities of individual stars in these streams, then

$$p=\sum v'^2+\sum v''^2 \text{ in unit volume}$$
$$=\nu\overline{v^2}$$
$$q=\sum(\mathrm{U}'+u')^2+\sum(-\mathrm{U}''+u'')^2=\nu'\mathrm{U}'^2+\nu''\mathrm{U}''^2+\nu\overline{u^2}.$$

In these equations we have $\nu' + \nu'' = \nu$, $\nu' U' = \nu'' U''$ and $\overline{u^2} = \overline{v^2}$, so that

$$\frac{q-p}{\nu} = \frac{\nu' U'^2 + \nu'' U''^2}{\nu} = U' U''.$$

If h_1, h_2 are the semiaxes of the Schwarzschild velocity ellipsoid at any point, we have

$$\frac{h_2^2}{h_1^2} = \frac{\Sigma\Theta^2}{\Sigma Z^2} = \frac{q}{p}.$$

7. In the neighbourhood of our sun we may take approximately $h_1/h_2 = 0{\cdot}577$, giving $h_2^2/h_1^2 = 3{\cdot}00$, or

$$\frac{q-p}{\nu} = \frac{p}{\nu} \times 2{\cdot}00.$$

In Table II. this relation is seen to be satisfied at $\varpi = 1090$ parsecs, at which $p/\nu = 14{,}000$ and $(q-p)/\nu = 28{,}000$. On the two-drift hypothesis, the numbers of stars in the two drifts are found to be almost exactly in the ratio 3 : 2, and the relative velocity of the two drifts is 40 kms. a second. Thus $U' = 16$ kms. a second and $U'' = 24$ kms. a second, giving

$$U'U'' = 384 \times 10^{10} \text{ C.G.S. units.}$$

We may notice in passing that the value of $U'U''$ is not very sensitive to the ratio in which the stars are supposed divided between the two streams. If we had supposed the streams to be equal, we should have had $U' = U'' = 20$ kms. a second, and the value of $U'U''$ would have altered only to 400×10^{10}.

We have now found that the value of $(q-p)/\nu$ in Kapteyn units is 28,000 and in C.G.S. units is 384×10^{10}. Hence, as in § 4, we must have

$$384 \times 10^{10} = \frac{\gamma M}{l} \times 28{,}000,$$

whence, on putting $\gamma = 6{\cdot}66 \times 10^{-8}$, $l = 3{\cdot}08 \times 10^{18}$, we find

$$M = 6{\cdot}34 \times 10^{33} \text{ grammes,}$$

which is the value which has been used throughout the paper, and has already been inserted in the values of Θ and Z given in Tables I. and II.

This value, being 3·2 times the mass of the sun, is obviously too large to represent the mass of a single star. The average mass of binary systems is found to be about 1·6 times the sun's mass, but these are mainly binaries of long period whose origin must probably be traced to separate nuclei in the original nebulæ, and whose combined mass may therefore be expected to be twice that of the average single star. Assuming the average star to have a mass of 0·8 times the sun's mass, our value of M must be interpreted as showing that if the stars are in a steady state, there must be about three dark stars in the universe to every bright star.

8. Subject to our being able to assume the existence of this very appreciable amount of dark matter, it appears that the observed motions

of the stars can be explained if, and only if, we are free to assume our sun to be at a distance of 1090 parsecs from the centre of the galaxy. This figure of 1090 parsecs cannot be regarded as having any great degree of accuracy, for we have made use of Kapteyn's observational distribution of density, which was derived only on the supposition that the sun is approximately at the centre of the system. The true value of the sun's distance from the centre can only be obtained as the result of a more refined investigation of stellar density, of which the need is fully recognised by Professor Kapteyn, in which full account will be taken from the outset of the excentric position of the sun.

9. Without carrying out this investigation in detail, it is easy to discover the general nature of the change to be expected in our estimate of 1090 parsecs. In place of the quantities z, ϖ, ν, p, q, M so far used, let us transform to new quantities z', ϖ', ν', p', q', M′ connected with the foregoing by the relations

$$z' = k_1 z, \quad \varpi' = k_1 \varpi, \quad \nu' = k_2 r, \quad p' = k_3 p, \quad q' = k_3 q, \quad \mathrm{M}' = k_4 \mathrm{M}$$

where k_1, k_2, k_3, k_4 are constants to which a value will shortly be assigned. Transformed to these quantities, our fundamental equations (9) and (10) become

$$\frac{\partial p'}{\partial z'} = \frac{k_1 k_3}{k_2^2 k_4} \nu' \frac{\partial \mathrm{V}'}{\partial z'}; \qquad \frac{\partial p'}{\partial \varpi'} + \frac{p' - q'}{\varpi'} = \frac{k_1 k_3}{k_2^2 k_4} \nu' \frac{\partial \mathrm{V}'}{\partial \varpi'}.$$

Thus the fundamental equations transform back to themselves if $k_1 k_3 = k_2^2 k_4$. If this relation is satisfied between the four k's there will be a solution in z', ϖ', etc., corresponding exactly to the solution we have already found in z, ϖ, etc. To what extent is it possible that this new solution should represent the observed facts after allowing for the excentric position of the sun?

The new solution gives a distribution of mass which is of course precisely similar to the old, except for proportional changes in length, mass, and density. To account for the observed ratio of axes of the Schwarzschild ellipsoid, our sun must be at the point in the new solution corresponding to $\varpi = 1090$ parsecs in the old solution; this will be given by $\varpi' = 1090$. To account for the observed density of stars near our sun we must have $\nu' = 0{\cdot}0451$ in the new solution as against $\nu = 0{\cdot}168$ in the old, so that $k_2 = 2{\cdot}68$. To account for the observed actual velocity of the stars in kilometres per second we must have $p'\mathrm{M}'/\nu' = p\mathrm{M}/\nu$, so that we must have $k_3 k_4 = k_2$. The new solution would give outer shells of the same shape as those of the old solution, but to make the observed density accord with observation a suitable value of k_1 must be chosen. Thus at $\varpi' = 6960$ the new solution gives $\nu = {\cdot}000715 \times 2{\cdot}68 = {\cdot}001910$, which observation shows to correspond to a distance of $\varpi = 4500$. Thus we must take $k_1 = 6960 \div 4500 = 1{\cdot}55$, and it will be found that this value of k_1 together with the previously found value of $k_2 = 2{\cdot}68$ gives a new solution which represents the observed distribution of stars in the outer layers with tolerable accuracy. Eliminating k_3 from the two relations $k_1 k_3 = k_2^2 k_4$ and $k_3 k_4 = k_2$ already found, we obtain $k_4^2 = k_1/k_2$, so that $k_4 = 0{\cdot}76$.

Thus in the new solution the sun's position ($\varpi' = 1090$ parsecs)

will be $\varpi = 700$ parsecs, and the average mass M will be $4{\cdot}8 \times 10^{33}$ grammes. Taking the average mass of a single star as 0·8 times the sun's mass, this indicates two dark stars to each bright star. In the new solution the stellar density at the sun's position ($\varpi = 700$ parsecs) will be 0·0451, the value assumed by Kapteyn, and the motion, as regards both star-streaming and peculiar motions of the stars, will be that actually observed in the neighbourhood of the sun. The new solution will represent Kapteyn's observational material well enough as regards the outer shells of stars, but the agreement with observation as regards the inner shells needs careful discussion. For reasons which will be found in his own paper, Kapteyn concludes that the greatest value of ϖ which can be reconciled with observation is in the neighbourhood of 700 parsecs. Thus our estimate of ϖ is just about at the limit permitted by observation. If the estimate had fallen entirely outside the limit, we should of course have had to conclude that the stars had not yet attained a steady state of the type assumed in this paper.

In any case we may notice that our estimate of $\varpi = 700$ parsecs places the centre of the galaxy right outside the system of the B stars as determined by Charlier. This leads us to regard the B stars rather as a moving star-cluster than as an integral part of the galactic system in steady motion with the rest of this system. We shall return to a consideration of this point of view in the next paper.

The Dynamics of Moving Clusters.

By J. H. Jeans.

1. In a recent "Research on Moving Clusters," * N. H. Rasmuson has provided material for testing a suggestion I put forward some years ago † that a moving cluster ought to assume a flattened shape as a consequence of encounters with other showers of stars. In particular, if the stars met by the cluster are moving at random the cluster ought to experience a flattening in the direction of its own motion. Certain of the clusters studied by Rasmuson (the Ursa Major cluster and that in Perseus, to quote his own instances) show a flattening of the type predicted, but others (*e.g.* the Scorpio-Centaurus cluster) show exactly the opposite effect.

In my investigation the forces acting on a single star of a moving cluster were supposed divided into three parts :

(*a*) The forces acting from the remaining stars of the cluster ;
(*b*) The forces from stars which are undergoing an "encounter" with this cluster, in the sense of the forces between single stars being of appreciable amount ;
(*c*) The forces from all the remaining stars of the universe.

* *Lund Meddel.*, 2, No. 26. † *M.N.*, 76 (1916), 552.

Forces (*c*) were supposed to be constant over the extent of a moving cluster, so that they merely accelerated the cluster as a whole without changing its shape. Forces (*a*) were neglected, and forces (*b*) were shown to result in the predicted flattening in the direction of motion. The object of this present note is to examine how far this prediction needs modification when forces (*a*) and (*b*) are taken fully into account.

2. To study forces (*c*), let us follow Kapteyn in assuming the stars of the universe to be arranged in spheroidal layers, having the galactic plane as plane of symmetry. At any point the resultant force from all external shells is zero, so that the whole gravitational force is the attraction of the internal shells. In regions near to the centre of the universe, let us suppose these shells to have a uniform density ρ, then the force will be deducible from a potential

$$V = C - A(x^2 + y^2) - Bz^2 \quad . \quad . \quad . \quad . \quad (1)$$

where A, B, C are constants, the galactic plane is the plane $z = 0$, and the galactic axis is the axis of z.

Poisson's equation $\nabla^2 V = -4\pi\rho$ at once provides the relation $2A + B = 4\pi\rho$. The ratio of B to A depends only on the ratio of the semiaxes of the spheroids, for which Kapteyn assumes the value $a/b = 5{\cdot}102$. With this value we readily find $B = 6{\cdot}31A$, so that $A = 0{\cdot}755\rho$, $B = 4{\cdot}78\rho$.

At the centroid x, y, z of the cluster the components of force are $-2Ax$, $-2Ay$, $-2Az$; at an adjacent point $x+\xi$, $y+\eta$, $z+\zeta$ they are $-2A(x+\xi)$, $-2A(y+\eta)$, $-2B(z+\zeta)$. Thus a star whose co-ordinates relative to the centroid are ξ, η, ζ will experience accelerations $-2A\xi$, $-2A\eta$, $-2B\zeta$, and if forces (*c*) were alone operative its equations of motion referred to axes moving with the centroid of the cluster would be

$$\ddot{\xi} = -2A\xi, \quad \ddot{\eta} = -2A\eta, \quad \ddot{\zeta} = -2B\zeta \quad . \quad . \quad . \quad (2)$$

Since A and B are both positive, it appears that each star's motion in the cluster will be compounded of three simple harmonic oscillations, two parallel to the plane of the galaxy of period $2\pi/(2A)^{\frac{1}{2}}$, and one perpendicular to this plane of period $2\pi/(2B)^{\frac{1}{2}}$. Let us assume for the density of matter in the inner regions of the galaxy a mass of $4{\cdot}8 \times 10^{33}$ gms. per 30 cubic parsecs,* or, in gravitational units, $\rho = 3{\cdot}65 \times 10^{-29}$, then the periods of the oscillations parallel to the galaxy are found to be 26,400,000 years, and that of the perpendicular oscillations is found to be 10,700,000 years.

Still neglecting forces (*a*), equations (2) will determine the motion of a star in the cluster except at those instants at which it is in close encounter with another star. Suppose that at any instant, as the result of such an encounter, a star experiences a force per unit mass of components F_x, F_y, F_z. Then equations (2) must at such an instant be replaced by

$$\ddot{\xi} = -2A\xi + F_x, \text{ etc.} \quad . \quad . \quad . \quad . \quad (3)$$

* See paper immediately preceding this (p. 132).

The general solution of this equation is *

$$\xi = \alpha \cos pt + \beta \sin pt + \frac{1}{p}\int_{t'=0}^{t'=t} F_x \sin p(t-t')dt' \quad . \quad . \quad (4)$$

where $p^2 = 2A$ and α, β are constants determined by the condition that $\alpha \cos pt + \beta \sin pt$ represents the oscillation which was being executed by ξ at time $t = 0$. In the present application $t = 0$ will be taken to be the beginning of the life of the cluster. As we shall find that the cluster scatters as it grows older, we shall assume α and β to be negligible in comparison with the present amplitude of the oscillation of ξ. We shall further suppose the value of F_x to arise solely from distinct encounters, so that the integral on the right of equation (4) can be separated into a sum of integrals each representing a contribution from a single encounter. In view of the numerical values which have been obtained for the period of $\sin p(t-t')$, we may reasonably suppose this sine term to remain constant throughout the duration of a single encounter, and so remove it outside the sign of integration. In this way equation (4) reduces to

$$\xi = \frac{1}{p}\sum \sin p(t-t')\int F_x dt,$$

where $\int F_x dt$ is obviously the impulse per unit mass, or change of velocity, produced by a single encounter. Let us denote $\int F_x dt$, $\int F_y dt$, $\int F_z dt$ for any encounter by I_x, I_y, I_z. Clearly in any single encounter I_x is just as likely to be positive as negative, and there will be no correlation between the values of I_x for different encounters. Hence we find for the expectation of ξ^2,

$$\xi^2 = \frac{1}{p^2}\sum \sin^2 p(t-t')I_x^2 \quad . \quad . \quad . \quad . \quad (5)$$

If the cluster has lived through a period at least as great as a half period of its oscillation, we may replace $\sin^2 p(t-t')$ by its mean value, namely $\frac{1}{2}$, and equation (5) becomes

$$\xi^2 = \frac{1}{4A}\Sigma I_x^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Similarly, of course, we obtain

$$\eta^2 = \frac{1}{4A}\Sigma I_y^2, \quad \zeta^2 = \frac{1}{4B}\Sigma I_z^2 \quad . \quad . \quad (7a, b)$$

If a cluster experiences encounters with a shower of stars whose velocity relative to it is approximately along Ox, it is readily seen that ΣI_y^2 and ΣI_z^2 are equal, while ΣI_x^2 is small in comparison, so that η^2 and ζ^2 will be large in comparison with ξ^2 and the cluster will be flattened in the direction Ox. This is the effect found in my previous paper, but equations (6) and (7) now show that this effect must be compounded with another one arising out of the presence of the factors $\frac{1}{4A}$ and $\frac{1}{4B}$

* Rayleigh, *Theory of Sound*, 1, § 66.

in ξ^2, η^2, and ζ^2. Since B is quite large compared with A, this second effect represents a tendency to flatten out in the plane of $\xi\eta$, *i.e.* parallel to the galactic plane.

To illustrate, suppose that a cluster is approximately at rest and is bombarded by stars arriving equally from all directions. It is readily seen that ΣI_x^2, ΣI_y^2, and ΣI_z^2 must all be equal, whence we see that the mean values of ξ^2, η^2, and ζ^2 will be in the ratio B : B : A. Since $\sqrt{B/A} = 2{\cdot}51$, the cluster will extend two and a half times as far in each direction parallel to the galaxy as it does perpendicular to the galaxy.

3. In arriving at equations (6) and (7), the assumption was made that $\sin^2 p(t-t')$ was equally likely to have all values from 0 to 1. For a cluster which has lived only a short fraction of a single period of oscillation this assumption is unjustifiable. In such a case $\sin p(t-t')$ will be small and may, to a fair approximation, be put equal to $p(t-t')$. Equations (6) and (7) must now be replaced by

$$\xi^2 = \Sigma(t-t')^2 I_x^2, \quad \eta^2 = \Sigma(t-t')^2 I_y^2, \quad \zeta^2 = \Sigma(t-t')^2 I_z^2.$$

The tendency to flattening in the direction of motion is present precisely as before, but the tendency to flattening parallel to the galactic plane has disappeared entirely. Thus young clusters are flattened in their direction of motion only, but with the onset of old age a further flattening parallel to the galactic plane makes its appearance.

4. It remains to examine the effects of forces (a), the forces from the other stars of the cluster. To an approximation which will prove to be sufficient, let us assume the stars of the cluster themselves to have a density ρ' which is uniform throughout the cluster. The force on a star at ξ, η, ζ arising from the other stars of the cluster will now have components

$$-\tfrac{4}{3}\pi\rho'\xi, \quad -\tfrac{4}{3}\pi\rho'\eta, \quad -\tfrac{4}{3}\pi\rho'\zeta.$$

To obtain the complete equations of motion these terms must be added to the right-hand members of equation (3). It is at once seen that the previous analysis holds entirely, provided we replace A, B by $A+\frac{2}{3}\pi\rho'$, $B+\frac{2}{3}\pi\rho'$ respectively. These quantities are nearer to equality than A and B, so that the effect of the inclusion of forces (a) is to diminish the flattening parallel to the galaxy.

In all of the clusters considered by Rasmuson $\frac{2}{3}\pi\rho'$ is quite small in comparison with A and B, except in the Taurus cluster. In this particular cluster he finds a density of 14·85 stars per cubic siriometer, or say one star per 8 cubic parsecs. Assuming the cluster stars to have an average mass of 3×10^{33} grammes, we obtain a cluster density (in gravitational units) $\rho' = 8{\cdot}6 \times 10^{-29}$, which is more than double the density we have assumed for the main universe, $A+\frac{2}{3}\pi\rho' = 4{\cdot}5\rho'$; $B+\frac{2}{3}\pi\rho' = 8{\cdot}5\rho'$. Thus the galactic flattening is reduced from 2·5 to 1·4, which is, in point of fact, about the degree of flattening found by Rasmuson in the Taurus cluster.

5. Apart from the Taurus cluster, Rasmuson has studied the distributions in space of four clusters. Three of these (Perseus, Scorpio-Centaurus, and 61 Cygni) exhibit a flattening parallel to the galactic plane, while two (Ursa Major and Perseus) show a flattening in the

direction of motion. In each of the three cases of flattening parallel to the galaxy the observed ratio of flattening is less than the limiting maximum (2·51) predicted by our theory, its greatest value (Scorpio-Centaurus) being 2·34.

It may here be remarked that the arrangement of the B-type stars, as determined by Charlier,* conforms closely to the arrangement to be expected on the hypothesis that they constitute a cluster moving with a low velocity. On account of their low velocity we should expect no flattening in their direction of motion, so that we should anticipate flattening parallel to the galactic plane and circular cross-sections parallel to this plane, which is what Charlier finds. On account of their sparseness, the ratio of flattening might be expected to approximate to the theoretical maximum 2·51; Charlier finds a ratio of flattening somewhat in excess of this, namely, 2·8.

The hypothesis that the B-type stars form a moving cluster, which we were led to contemplate in the previous paper from a consideration of their excentric position with regard to the main structure of the galaxy, has in its favour the observed circumstance that these stars appear not to form part of either of the two main star streams, forming rather a third distinct drift. If the stars form a cluster, this cluster must be in process of expansion, and equations (6) and (7) provide the means of calculating the rate of expansion.

Since the cluster is approximately at rest in space, we may suppose it bombarded equally in all directions by extraneous stars. Thus we may put

$$\Sigma I_x^2 = \Sigma I_y^2 = \Sigma I_z^2 = \tfrac{1}{3}\Sigma I^2.$$

In a previous paper† the value of I at a single encounter was found to be $2m/V\sigma$, where σ was the distance of nearest approach, m the mass of the encountering star, and V the relative velocity. With a star density ν the value of ΣI^2 accruing from encounters in time t is found to be

$$\frac{8\pi m^2\nu}{V}\log\left(\frac{\sigma_2}{\sigma_1}\right)t,$$

so that equation (6) becomes

$$\frac{d\xi^2}{dt} = \frac{1}{12A}\cdot\frac{8\pi m^2\nu}{V}\log\left(\frac{\sigma_2}{\sigma_1}\right).$$

On taking $\log(\sigma_2/\sigma_1) = 13{\cdot}42$ as in the previous paper, and $A = 0{\cdot}755\rho = 0{\cdot}755m\nu$ from the present paper, this becomes

$$\frac{d\xi^2}{dt} = 37{\cdot}3\frac{m}{V}.$$

Remembering that the most massive stars, and encounters with the smallest relative velocity, are most effective in increasing ξ^2, we may take $m = 3\times10^{33}$ gms. ($= 2\times10^{26}$ in gravitational units) and $V = 5\times10^5$ cms. a second, giving $\xi^2 = 1{\cdot}5\times10^{22}t$. For a point for which $\xi = 200$ parsecs ($= 6{\cdot}2\times10^{20}$ cms.) the expectation of recessive velocity is only

* *Lund Meddel.*, 2, No. 14. † *M.N.*, 76 (1916), 555.

about 12 cms. a second, or about 1 parsec in 8000 million years, representing a very slow rate of growth.

6. The dynamics of a cluster which is either entirely outside the galactic system, or in an outer region in which stars are very sparse, is considerably different. If x, y, z again denote the centroid of a cluster, the potential at a point $x+\xi$, $y+\eta$, $z+\zeta$ of all stars outside the cluster may be expanded in the form

$$V = V_0 + \xi\frac{\partial V}{\partial x} + \eta\frac{\partial V}{\partial y} + \zeta\frac{\partial V}{\partial z} + \tfrac{1}{2}\left(\xi^2\frac{\partial^2 V}{\partial x^2} + 2\xi\eta\frac{\partial^2 V}{\partial x \partial y} + \ldots\right).$$

Thus as regards the motion of a star inside the cluster, the effect of external stars and systems of stars may be regarded as being derived from a potential

$$\tfrac{1}{2}\left(\xi^2\frac{\partial^2 V}{\partial x^2} + 2\xi\eta\frac{\partial^2 V}{\partial x \partial y} + \ldots\right).$$

This quadratic expression may be transformed to its principal axes, say ξ', η', ζ', and will assume the form

$$\tfrac{1}{2}(\alpha\xi'^2 + \beta\eta'^2 + \gamma\zeta'^2),$$

where, from a well-known invariant property,

$$\alpha + \beta + \gamma = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Since the sum of α, β, and γ is zero, one at least of these coefficients, say α, must be positive.

Suppose the point ξ, η, ζ to be at a distance r from the centroid of the cluster. Let the total mass of all the cluster stars within a distance r of the centroid be M_r, and put this equal to $\frac{4}{3}\pi\rho r^3$, so that ρ is the mean density of stars up to a distance r from the centroid of the cluster. To a fair approximation, especially for a "globular cluster," we may assume that a star at distance r from the centre experiences from all the other stars of the cluster an attractive force M_r/r^2 towards the centre. The components of this force along the axes ξ', η', ζ' will be $-\frac{4}{3}\pi\rho\xi'$, $-\frac{4}{3}\pi\rho\eta'$, $-\frac{4}{3}\pi\rho\zeta'$. Thus the equations of motion, relative to the centroid, of a star at ξ', η', ζ' will be

$$\ddot{\xi}' = (\alpha - \tfrac{4}{3}\pi\rho)\xi' \quad . \quad . \quad . \quad . \quad (8)$$

$$\ddot{\eta}' = (\beta - \tfrac{4}{3}\pi\rho)\eta' \quad . \quad . \quad . \quad . \quad (9)$$

$$\ddot{\zeta}' = (\gamma - \tfrac{4}{3}\pi\rho)\zeta' \quad . \quad . \quad . \quad . \quad (10)$$

We have already seen that one of the quantities α, β, γ, say α, must be positive. The maximum value of ρ in the cluster will be the value ρ_0 at the centre. Now if α is greater than $\frac{4}{3}\pi\rho_0$, equation (8) is of the form $\ddot{\xi}' = \kappa^2\xi'$ where κ^2 is positive, and the solution is of the form

$$\xi' = C_1 e^{\kappa t} + C_2 e^{-\kappa t}.$$

Thus, after an initial period in which the second term disappears, the diameter of the cluster in the direction of the axis of ξ' will

increase exponentially with the time. For a somewhat sparse cluster on the confines of the galactic system α and ρ in gravitational units will each be of the order of 10^{-30}, which results in an e-fold increase in about 10 million years.

If ρ_0 is so great that α is less than $\frac{4}{3}\pi\rho$, there will be a critical value of ρ given by $\alpha = \frac{4}{3}\pi\rho$, and the expansion we have found will occur only outside the sphere of radius r which corresponds to the critical value of ρ. But the essential point is that whatever the constitution of the cluster there must be comparatively rapid dissipation of its outer shells at least.

Charlier's study of stellar clusters* has shown how closely all clusters, except globular clusters, are confined to the neighbourhood of the galactic plane. It is readily seen that in or near the galactic plane the principal axis ξ', for which $\partial^2 V/\partial\xi'^2$ is positive, will lie approximately parallel to the galactic plane, so that the clusters outside the dense central regions of the galaxy ought to spread with comparative rapidity in this plane.

7. It is perhaps worth considering for a moment the hypothesis that our whole galactic system may have been formed simply out of a collection of moving clusters and the *débris* of disintegrated clusters. We can imagine a system of clusters moving in the galactic plane and spreading in that plane under their mutual gravitational forces. When a pair of clusters pass through one another they will also spread perpendicular to the galactic plane. The potential energy of each is increased by this spreading, the necessary energy being provided by a decrease of the energy of translation of the clusters as a whole, so that finally we can imagine the clusters to be reduced to relative rest in a configuration in which they are all intermingled.

Let W_1, T_1 denote the potential and internal kinetic energy of a single cluster before mingling with the rest, and let T_1 be its energy of translation relative to the centroid of all the clusters. Let the whole system of clusters mingle into a single cluster of potential energy W_0 and internal kinetic energy T_0. Then the equation of energy provides the relation

$$W_0 + T_0 = \Sigma(W_1 + T_1 + \mathsf{T}_1) \quad . \quad . \quad . \quad (11)$$

If each constituent cluster were initially in a state of steady motion

$$2T_1 + W_1 = 0, \text{ etc.} \quad . \quad . \quad . \quad . \quad (12)$$

while if the final cluster is also supposed in a steady state

$$2T_0 + W_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By simple algebra we deduce the relation

$$W_0 = \Sigma(W_1 + 2\mathsf{T}_1).$$

Thus it appears that in the act of combination an amount of energy equal to twice the original total energy of translation of all the clusters is changed into potential energy. With even moderate values for the

* *Lund Meddel.*, 2, No. 19.

translational velocities of the clusters it is found that the resulting expansion of the final cluster may be enormous. There is no reason why our greatly extended galactic system should not have been formed by the coming together of a number of quite compact clusters, of which the only recognisable remains are now to be found in the "moving clusters." If we imagine these to have been born in the galactic plane, we obtain a ready explanation of the circumstance that the moving star clusters now all move approximately parallel to the galactic plane. The hypothesis has another feature of interest. Many circumstances, and especially the distribution of orbits of binary stars, suggest that the stars of our system must at some past time have been more closely packed than they now are. The hypothesis provides a dynamical mechanism by which the primæval close packing may have been changed into the present sparse distribution of stars.

The Existence Theorem for certain Periodic Orbits in the Restricted Problem of Three Bodies. By W. M. H. Greaves, B.A., Isaac Newton Student in the University of Cambridge.

(*Communicated by Professor H. F. Baker.*)

In chapter ix. of Moulton's *Periodic Orbits* * Buck investigates certain periodic orbits which an infinitesimal body may describe when attracted according to the Newtonian Law by two finite bodies revolving in circles round each other under the influence of their mutual gravitational attraction. The vertices of the two equilateral triangles described in the plane of motion of the two finite bodies on the line joining them as base are positions of relative equilibrium of the infinitesimal body, and Buck investigates those motions of this body in the vicinity of the equilateral triangle points which are periodic with respect to the two finite bodies, so that configurations will recur.

In this paper the *existence* of two families † of periodic orbits in the plane of motion of the two finite masses is established in a rather different way. The demonstration is similar in character to the existence proofs given by Poincaré,‡ and may perhaps be of interest.

Let S and J denote the two larger bodies, A the body of infinitesimal mass.

J and S describe circles round each other under the influence of their mutual gravitational attraction.

Let B be the vertex of an equilateral triangle described on SJ as base in the plane of motion. (We shall suppose that the motion of A is confined to the plane of motion of S and J.)

* Carnegie Institute of Washington, Publication No. 161.

† The families arising from the first and second generating solutions of Buck's paper. One of the families is that investigated by Brown in *Monthly Notices R.A.S.*, **71**, 5, March 1911.

‡ *Méthodes nouvelles*, t. i. ch. iii.

Choose the unit of mass so that the mass of S is unity. Let m be the mass of J.

Choose the unit of length so that the length of SJ is unity. That being done, choose the unit of time so that the numerical value of the constant of gravitation is unity.

Let n be the L^r velocity of SJ, so that

$$n^2 = 1 + m \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Take moving axes Bx, By along and perpendicular to SB produced.

Let the co-ordinates of A referred to this system of axes be ϵx and ϵy where ϵ is an arbitrary parameter.

The equations of motion are

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial V}{\partial x} \\ \ddot{y} + 2n\dot{x} &= \frac{\partial V}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)^*$$

where

$$V = \sum_{r=2}^{r=\infty} \frac{\epsilon^{r-2}}{\lfloor r} \{ p_{r0}x^r + {}^rc_1 p_{r-1,\,1} x^{r-1}y + {}^rc_2 p_{r-2,\,2} x^{r-2}y^2 + \ldots + p_{0r}y^r \} \quad (3)$$

the values of the first nine coefficients being

$$\begin{aligned} p_{10} &= p_{01} = 0 & p_{30} &= -6 + \frac{21m}{8} \\ p_{20} &= 3(1 + \tfrac{1}{4}m) & p_{03} &= -\frac{9\sqrt{3}m}{8} \\ p_{11} &= \frac{3\sqrt{3}m}{4} & p_{21} &= -\frac{3\sqrt{3}m}{8} \\ p_{02} &= \tfrac{9}{4}m & p_{12} &= 3(1 - \tfrac{11}{8}m). \end{aligned}$$

When $\epsilon = 0$ the equations (2) become

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= p_{20}x + p_{11}y \\ \ddot{y} + 2n\dot{x} &= p_{11}x + p_{02}y \end{aligned}$$

and the general solution of these equations is found to be

$$\left.\begin{aligned} x &= A\cos\lambda_1 t + B\sin\lambda_1 t + C\cos\lambda_2 t + D\sin\lambda_2 t \\ y &= A'\cos\lambda_1 t + B'\sin\lambda_1 t + C'\cos\lambda_2 t + D'\sin\lambda_2 t \end{aligned}\right\} \quad . \quad (4)$$

where λ_1^2 and λ_2^2 are the roots of the quadrate in λ^2

$$(\lambda^2 + p_{20})(\lambda^2 + p_{02}) = p_{11}^2 + 4n^2\lambda^2 \quad . \quad . \quad . \quad (5)$$

We shall assume that $(1+m)^2 > 27m$, so that λ_1 and λ_2 are real and unequal. A, B, C′, and D′ will be taken as the fundamental constants

* These equations are easily derived from the equations of motion used by Brown.

of integration, and the constants A′, B′, C, and D will be given by the relations

$$\left.\begin{aligned} A' &= q_1 B - q_2 A & C &= -r_1 D' - r_2 C' \\ B' &= -q_1 A - q_2 B & D &= r_1 C' - r_2 D' \end{aligned}\right\} \quad . \quad . \quad (6)$$

$$\left.\begin{aligned} q_1 &= \frac{2n\lambda_1}{\lambda_1^2 + p_{02}} & r_1 &= \frac{2n\lambda_2}{\lambda_2^2 + p_{20}} \\ q_2 &= \frac{p_{11}}{\lambda_1^2 + p_{02}} & r_2 &= \frac{p_{11}}{\lambda_2^2 + p_{20}} \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

Now, writing $C' = D' = 0$, $A = a$, $B = b$, $A = a'$, $B = b'$, the solution (4) reduces to the periodic solution

$$\left.\begin{aligned} x &= a\cos\lambda_1 t + b\sin\lambda_1 t \\ y &= a'\cos\lambda_1 t + b'\sin\lambda_1 t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4')$$

Here a and b are arbitrary, and a' and b' are given by

$$\left.\begin{aligned} a' &= q_1 b - q_2 a \\ b' &= -q_1 a - q_2 b \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

The initial values of xy, $\dot{x}\dot{y}$ are

$$\begin{aligned} x &= a & \dot{x} &= \lambda_1 b \\ y &= a' & \dot{y} &= \lambda_1 b' \end{aligned}$$

The period $= \dfrac{2\pi}{\lambda_1} = T$ say.

Now suppose ϵ is no longer zero, and the initial values are

$$\left.\begin{aligned} P &= x = a + B_1 \\ Q &= y = a' + B_2 \\ R &= \dot{x} = \lambda_1 b + B_3 \\ S &= \dot{y} = \lambda_1 b' + B_4 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

and suppose at time $T + \tau$ the values are

$$\left.\begin{aligned} x &= a + B_1 + \Psi_1 \\ y &= a' + B_2 + \Psi_2 \\ \dot{x} &= \lambda_1 b + B_3 + \Psi_3 \\ \dot{y} &= \lambda_1 b' + B_4 + \Psi_4 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

From Poincaré's extension of Cauchy's theorem* it follows that Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are holomorphic functions of B_1, B_2, B_3, B_4, τ, and ϵ.

If we can determine B_1, B_2, B_3, B_4, and τ as functions of ϵ, vanishing when $\epsilon = 0$, so that the equations $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ are satisfied, we shall have found a set of initial values leading to a periodic solution with period $T + \tau$.

* *Méthodes nouvelles*, t. i. ch. ii. pp. 51 *et seq.*

Now these four equations are not independent. For it is easy to see that the equations (2) have the integral $\dot{x}^2 + \dot{y}^2 = 2V - C$ where C is a constant.

Hence if we put $F = V - \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$, F is constant throughout the motion.

F is a function of x, y, $\dot{x}$, $\dot{y}$, and may be written $F(x\ y,\ \dot{x}\ \dot{y})$.

On account of the constancy of F we have

$$F(a + B_1,\ a' + B_2,\ \lambda_1 b + B_3,\ \lambda_1 b' + B_4) = F(a + B_1 + \Psi_1,\ a' + B_2 + \Psi_2,\ \lambda_1 b + B_3 + \Psi_3,\ \lambda_1 b + B_4 + \Psi_4),$$

so that if three of the quantities Ψ_1, Ψ_2, Ψ_3, Ψ_4 vanish, so does the fourth.

Hence it will be sufficient to consider the equations

$$\Psi_1 = \Psi_2 = \Psi_3 = 0 \quad . \quad . \quad . \quad . \quad (11)$$

We want to choose B_1, B_2, B_3, B_4, and τ so as to satisfy (11).

Now, if we put any two of these five quantities equal to zero, and if the jacobian of Ψ_1, Ψ_2, Ψ_3 with respect to the remaining three does not vanish when $B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0$, then it follows* that for sufficiently small values of ϵ we can find these three quantities as holomorphic functions of ϵ vanishing when $\epsilon = 0$ so as to satisfy (13), thereby giving a periodic solution.

We shall find that if we put $\tau = 0$ and one of the B's zero, the jacobian of the Ψ's with respect to the remaining three B's does vanish, so that it is impossible to demonstrate in this way the existence of periodic solutions of period T.

But we shall find that it is possible to put two of the B's equal to zero and to prove that the jacobian of the Ψ's with respect to the other two B's and τ does not vanish when $B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0$, so that we can demonstrate the existence of periodic solutions for sufficiently small values of ϵ whose periods are nearly equal to T, *i.e.* differing from T by quantities of the same order of magnitude as ϵ.

Furthermore, in evaluating our jacobians we may put $\epsilon = 0$ from the first, since no derivatives with respect to ϵ enter into the elements.

Suppose, then, that $\epsilon = 0$ and the initial values are

$$\left.\begin{aligned} x &= P = a + B_1 \\ y &= Q = a' + B_2 \\ \dot{x} &= R = \lambda_1 b + B_3 \\ \dot{y} &= S = \lambda_1 b' + B_4 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

and that the solution corresponding to these initial values is

$$\left.\begin{aligned} x &= A \cos \lambda_1 t + B \sin \lambda_1 t + C \cos \lambda_2 t + D \sin \lambda_2 t \\ y &= A' \cos \lambda_1 t + B' \sin \lambda_1 t + C' \cos \lambda_2 t + D' \sin \lambda_2 t \end{aligned}\right\} \quad . \quad (13$$

where A', B', C, and D are connected with A, B, C', and D' by the relations (6).

* Goursat, *Cours d'analyse*, t. i. ch. ix. pp. 479–481.

From the initial conditions we have, using (6),

$$\begin{aligned} A \qquad\qquad & -r_2C' \quad -r_1D' \quad = P \\ -q_2A \quad +q_1B \quad & +C' \qquad\qquad = Q \\ \lambda_1 B \quad & +\lambda_2 r_1 C' - \lambda_2 r_2 D' = R \\ -\lambda_1 q_1 A - \lambda_1 q_2 B \quad & +\lambda_2 D' \quad = S \end{aligned}$$

Solving these for A, B, C′, and D′, we find

$$\left.\begin{aligned} A &= k_{11}P + k_{12}Q + k_{13}R + k_{14}S \\ B &= k_{21}P + k_{22}Q + k_{23}R + k_{24}S \\ C' &= k_{31}P + k_{32}Q + k_{33}R + k_{34}S \\ D' &= k_{41}P + k_{42}Q + k_{43}R + k_{44}S \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

and we then obtain from (6)

$$\left.\begin{aligned} A' &= g_{11}P + g_{12}Q + g_{13}R + g_{14}S \\ B' &= g_{21}P + g_{22}Q + g_{23}R + g_{24}S \\ C &= g_{31}P + g_{32}Q + g_{33}R + g_{34}S \\ D &= g_{41}P + g_{42}Q + g_{43}R + g_{44}S \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

where

$$\left.\begin{aligned} g_{1s} &= q_1 k_{2s} - q_2 k_{1s} \\ g_{2s} &= -q_1 k_{1s} - q_2 k_{2s} \\ g_{3s} &= -r_1 k_{4s} - r_2 k_{3s} \\ g_{4s} &= r_1 k_{3s} - r_2 k_{4s} \end{aligned} \qquad (s = 1, 2, 3, 4)\right\} \quad . \quad . \quad (16)$$

From (16) we deduce the reciprocal relations

$$\left.\begin{aligned} k_{1s} &= q_1' g_{2s} - q_2' g_{1s} \\ k_{2s} &= -q_1' g_{1s} - q_2' g_{2s} \\ k_{3s} &= -r_1' g_{4s} - r_2' g_{3s} \\ k_{4s} &= r_1' g_{3s} - r_2' g_{4s} \end{aligned} \qquad (s = 1, 2, 3, 4)\right\} \quad . \quad . \quad (17)$$

where

$$\left.\begin{aligned} q_1' &= -\frac{q_1}{q_1^2 + q_2^2} & r_1' &= -\frac{r_1}{r_1^2 + r_2^2} \\ q_2' &= \frac{q_2}{q_1^2 + q_2^2} & r_2' &= \frac{r_2}{r_1^2 + r_2^2} \end{aligned}\right\} \quad . \quad . \quad (18)$$

It is found on calculation that

$$k_{pq} = \frac{h_{pq}}{\Delta}$$

where

$$\left.\begin{aligned}
&\Delta = q_1r_1(\lambda_1^2+\lambda_2^2)+(2q_2r_2-1)\lambda_1\lambda_2-(q_1^2+q_2^2)(r_1^2+r_2^2)\lambda_1\lambda_2\\
&h_{11}=q_1r_1\lambda_2^2+(q_2r_2-1)\lambda_1\lambda_2 \qquad h_{31}=(q_1^2+q_2^2)r_2\lambda_1\lambda_2-q_2\lambda_1\lambda_2\\
&h_{12}=\lambda_1\lambda_2\{q_2(r_1^2+r_2^2)-r_2\} \qquad h_{32}=q_1r_1\lambda_1^2+q_2r_2\lambda_1\lambda_2-\lambda_1\lambda_2\\
&h_{13}=q_1\lambda_2r_2-q_2\lambda_1r_1 \qquad h_{33}=q_1\lambda_2-(q_1^2+q_2^2)r_1\lambda_1\\
&h_{14}=q_1\lambda_2(r_1^2+r_2^2)-r_1\lambda_1 \qquad h_{34}=q_1r_2\lambda_2-q_2r_1\lambda_1\\
&h_{21}=q_2r_1\lambda_2^2-q_1r_2\lambda_1\lambda_2 \qquad h_{41}=(q_1^2+q_2^2)r_1\lambda_1\lambda_2-q_1\lambda_1^2\\
&h_{22}=\lambda_2^2r_1-\lambda_1\lambda_2q_1(r_1^2+r_2^2) \qquad h_{42}=\lambda_1\lambda_2q_2r_1-\lambda_1^2q_1r_2\\
&h_{23}=q_1r_1\lambda_1+(q_2r_2-1)\lambda_2 \qquad h_{43}=(q_1^2+q_2^2)r_2\lambda_1-q_2\lambda_1\\
&h_{24}=q_2(r_1^2+r_2^2)\lambda_2-r_2\lambda_2 \qquad h_{44}=q_1r_1\lambda_2+(q_2r_2-1)\lambda_1
\end{aligned}\right\} \quad (19)$$

and it is easily verified that

$$\left.\begin{aligned}
g_{31}&=1-k_{11} & \lambda_2g_{41}&=-\lambda_1k_{21}\\
g_{32}&=-k_{12} & \lambda_2g_{42}&=-\lambda_1k_{22}\\
g_{33}&=-k_{13} & \lambda_2g_{43}&=1-\lambda_1k_{23}\\
g_{34}&=-k_{14} & \lambda_2g_{44}&=-\lambda_1k_{24}\\
g_{11}&=-k_{31} & \lambda_1g_{21}&=-\lambda_2k_{41}\\
-1+g_{12}&=-k_{32} & \lambda_1g_{22}&=-\lambda_2k_{42}\\
g_{13}&=-k_{33} & \lambda_1g_{23}&=-\lambda_2k_{43}\\
g_{14}&=-k_{34} & -1+\lambda_1g_{24}&=-\lambda_2k_{44}
\end{aligned}\right\} \quad . \quad . \quad (20)$$

At time $t=\mathrm{T}+\tau$ we have

$$\begin{aligned}
x&=a+\mathrm{B}_1+\Psi_1\\
y&=a'+\mathrm{B}_2+\Psi_2\\
\dot{x}&=\lambda_1b+\mathrm{B}_3+\Psi_3\\
\dot{y}&=\lambda_1b'+\mathrm{B}_4+\Psi_4.
\end{aligned}$$

Hence, from (13)

$$\left.\begin{aligned}
a+\mathrm{B}_1+\Psi_1&=\mathrm{A}\cos\lambda_1t'+\mathrm{B}\sin\lambda_1t'+\mathrm{C}\cos\lambda_2t'+\mathrm{D}\sin\lambda_2t'\\
a'+\mathrm{B}_2+\Psi_2&=\mathrm{A}'\cos\lambda_1t'+\mathrm{B}'\sin\lambda_1t'+\mathrm{C}'\cos\lambda_2t'+\mathrm{D}'\sin\lambda_2t'\\
\lambda_1b+\mathrm{B}_3+\Psi_3&=\lambda_1\mathrm{B}\cos\lambda_1t'-\lambda_1\mathrm{A}\sin\lambda_1t'+\lambda_2\mathrm{D}\cos\lambda_2t'-\lambda_2\mathrm{C}\sin\lambda_2t'\\
\lambda_1b+\mathrm{B}_4+\Psi_4&=\lambda_1\mathrm{B}'\cos\lambda_1t'-\lambda_1\mathrm{A}'\sin\lambda_1t'+\lambda_2\mathrm{D}'\cos\lambda_2t'-\lambda_2\mathrm{C}'\sin\lambda_2t'
\end{aligned}\right\} \quad (21)$$

where $t'=\mathrm{T}+\tau.$

Note that A B C D, A′ B′ C′ D′ involve B_1 B_2 B_3 B_4 linearly through P Q R and S.

From (21) we can at once find the derivatives of the Ψ's with respect to the B's and τ and then put $\mathrm{B}_1=\mathrm{B}_2=\mathrm{B}_3=\mathrm{B}_4=\tau=0$.

Note that in finding the derivatives with respect to the B's we can put $\tau=0$ at the outset, so that $t'=\mathrm{T}$, $\lambda_1t'=\lambda_1\mathrm{T}=2\pi$, and $\lambda_2t'=\frac{2\pi\lambda_2}{\lambda_1}=\theta$ say.

Hence for the purposes of finding the derivatives of the Ψ's with respect to the B's equations (21) may be replaced by

$$\left.\begin{aligned} a + B_1 + \Psi_1 &= A + C\cos\theta + D\sin\theta \\ a' + B_2 + \Psi_2 &= A' + C'\cos\theta + D'\sin\theta \\ \lambda_1 b + B_3 + \Psi_3 &= \lambda_1 B + \lambda_2 D\cos\theta - \lambda_2 C\sin\theta \\ \lambda_1 b' + B_4 + \Psi_4 &= \lambda_1 B' + \lambda_2 D'\cos\theta - \lambda_2 C'\sin\theta \end{aligned}\right\} . \quad (21')$$

where $$\theta = \frac{2\pi\lambda_2}{\lambda_1}.$$

From (21'), using (14) and (15) and remembering that $P = a + B_1$, $Q = a' + B_2$, $R = \lambda_1 b + B_3$, $S = \lambda_1 b' + B_4$, we have, when $B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0$,

$$\frac{\partial\Psi_1}{\partial B_1} = -1 + k_{11} + g_{31}\cos\theta + g_{41}\sin\theta$$

$$\frac{\partial\Psi_1}{\partial B_2} = k_{12} + g_{32}\cos\theta + g_{42}\sin\theta$$

$$\frac{\partial\Psi_1}{\partial B_3} = k_{13} + g_{33}\cos\theta + g_{43}\sin\theta$$

$$\frac{\partial\Psi_1}{\partial B_4} = k_{14} + g_{34}\cos\theta + g_{44}\sin\theta$$

$$\frac{\partial\Psi_2}{\partial B_1} = g_{11} + k_{31}\cos\theta + k_{41}\sin\theta$$

$$\frac{\partial\Psi_2}{\partial B_2} = -1 + g_{12} + k_{32}\cos\theta + k_{42}\sin\theta$$

$$\frac{\partial\Psi_2}{\partial B_3} = g_{13} + k_{33}\cos\theta + k_{43}\sin\theta$$

$$\frac{\partial\Psi_2}{\partial B_4} = g_{14} + k_{34}\cos\theta + k_{44}\sin\theta$$

$$\frac{\partial\Psi_3}{\partial B_1} = \lambda_1 k_{21} + \lambda_2 g_{41}\cos\theta - \lambda_2 g_{31}\sin\theta$$

$$\frac{\partial\Psi_3}{\partial B_2} = \lambda_1 k_{22} + \lambda_2 g_{42}\cos\theta - \lambda_2 g_{32}\sin\theta$$

$$\frac{\partial\Psi_3}{\partial B_3} = -1 + \lambda_1 k_{23} + \lambda_2 g_{43}\cos\theta - \lambda_2 g_{33}\sin\theta$$

$$\frac{\partial\Psi_3}{\partial B_4} = \lambda_1 k_{24} + \lambda_2 g_{44}\cos\theta - \lambda_2 g_{34}\sin\theta$$

$$\frac{\partial\Psi_4}{\partial B_1} = \lambda_1 g_{21} + \lambda_2 k_{41}\cos\theta - \lambda_2 k_{31}\sin\theta$$

$$\frac{\partial\Psi_4}{\partial B_2} = \lambda_1 g_{22} + \lambda_2 k_{42}\cos\theta - \lambda_2 k_{32}\sin\theta$$

$$\frac{\partial\Psi_4}{\partial B_3} = \lambda_1 g_{23} + \lambda_2 k_{43}\cos\theta - \lambda_2 k_{33}\sin\theta$$

$$\frac{\partial\Psi_4}{\partial B_4} = -1 + \lambda_1 g_{24} + \lambda_2 k_{44}\cos\theta - \lambda_2 k_{34}\sin\theta.$$

Now, put
$$\left.\begin{aligned} 1-\cos\theta &= u \\ \sin\theta &= v \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

From the above expressions we find, using (20),

$$\left.\begin{aligned} \frac{\partial\Psi_1}{\partial B_s} &= -g_{3s}u + g_{4s}v \\ \frac{\partial\Psi_2}{\partial B_s} &= -k_{3s}u + k_{4s}v \\ \frac{\partial\Psi_3}{\partial B_s} &= -\lambda_2(g_{4s}u + g_{3s}v) \\ \frac{\partial\Psi_4}{\partial B_s} &= -\lambda_2(k_{4s}u + k_{3s}v) \end{aligned}\right\} \quad (s = 1,\ 2,\ 3,\ 4) \quad . \quad . \quad (23)$$

From (23), using (17), we find,

$$\begin{aligned} r_2'\frac{\partial\Psi_1}{\partial B_s} + \frac{r_1'}{\lambda_2}\frac{\partial\Psi_3}{\partial B_s} &= u\{-r_2'g_{3s} - r_1'g_{4s}\} + v(r_2'g_{4s} - r_1'g_{3s}) \\ &= k_{3s}u - g_{4s}v \\ &= -\frac{\partial\Psi_2}{\partial B_s}. \end{aligned}$$

$\therefore \frac{\partial(\Psi_1\Psi_2\Psi_3)}{\partial(lmn)} = 0$, where l, m, and n are any three of the B's. So that, if we put $\tau = 0$ and one of the B's zero, the jacobian of the Ψ's with respect to the remaining three B's vanishes when $B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0$, and the method fails.

We cannot, then, demonstrate in this way the existence of periodic solutions of period T. It would appear from Buck's paper that such solutions do not exist.

Again, from (21),

$$\frac{\partial\Psi_1}{\partial\tau} = \lambda_1 B\cos\lambda_1 t' - \lambda_1 A\sin\lambda_1 t' + \lambda_2 D\cos\lambda_2 t' - \lambda_2 C\sin\lambda_2 t'$$

$$\frac{\partial\Psi_2}{\partial\tau} = \lambda_1 B'\cos\lambda_1 t' - \lambda_1 A'\sin\lambda_1 t' + \lambda_2 D'\cos\lambda_2 t' - \lambda_2 C'\sin\lambda_2 t'$$

$$\frac{\partial\Psi_3}{\partial\tau} = -\lambda_1^2 A\cos\lambda_1 t' - \lambda_1^2 B\sin\lambda_1 t' - \lambda_2^2 C\cos\lambda_2 t' - \lambda_2^2 D\sin\lambda_2 t'$$

$$\frac{\partial\Psi_4}{\partial\tau} = -\lambda_1^2 A'\cos\lambda_1 t' - \lambda_1^2 B'\sin\lambda_1 t' - \lambda_2^2 C'\cos\lambda_2 t' - \lambda_2^2 D'\sin\lambda_2 t'.$$

When $\tau = B_1 = B_2 = B_3 = B_4 = \epsilon = 0$, so that $\lambda_1 t' = \pi$ and $\lambda_2 t' = \theta$, these become

$$\frac{\partial\Psi_1}{\partial\tau} = \lambda_1 B + \lambda_2 D\cos\theta - \lambda_2 C\sin\theta$$

$$\frac{\partial\Psi_2}{\partial\tau} = \lambda_1 B' + \lambda_2 D'\cos\theta - \lambda_2 C'\sin\theta$$

$$\frac{\partial\Psi_3}{\partial\tau} = -\lambda_1^2 A - \lambda_2^2 C\cos\theta - \lambda_2^2 D\sin\theta$$

$$\frac{\partial\Psi_4}{\partial\tau} = -\lambda_1^2 A' - \lambda_2^2 C'\cos\theta - \lambda_2^2 D'\sin\theta.$$

But, since the B's are zero we have $C = D = C' = D' = 0$, $A = a$, $A' = a'$, $B = b$, and $B' = b'$.

Hence

$$\frac{\partial \Psi_1}{\partial \tau} = \lambda_1 b \qquad \frac{\partial}{\partial \tau}^2 = \lambda_1 b'$$
$$\frac{\partial \Psi_3}{\partial \tau} = -\lambda_1^2 a \qquad \frac{\partial \Psi_4}{\partial \tau} = -\lambda_1^2 a' \qquad \text{when } B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0 \quad (24)$$

Now let J denote the jacobian $\frac{\partial(\Psi_1 \Psi_2 \Psi_3)}{\partial(\tau B_2 B_3)}$ for $B_1 = B_2 = B_3 = B_4 = \tau = \epsilon = 0$.

We easily find

$$J = \lambda_1 b F + \lambda_1 b' G - \lambda_1^2 a H$$

where

$$F = \frac{\partial(\Psi_2 \Psi_3)}{\partial(B_2 B_3)} \qquad G = \frac{\partial(\Psi_3 \Psi_1)}{\partial(B_2 B_3)} \qquad H = \frac{\partial(\Psi_1 \Psi_2)}{\partial(B_2 B_3)}.$$

F, G, and H can be at once evaluated from (23). Using (16), they reduce to

$$F = \lambda_2 r_2 (u^2 + v^2)(k_{33} k_{42} - k_{32} k_{43})$$
$$G = \lambda_2 (u^2 + v^2)(r_1^2 + r_2^2)(k_{33} k_{42} - k_{43} k_{32})$$
$$H = -r_1 (u^2 + v^2)(k_{33} k_{42} - k_{43} k_{32}),$$

so that

$$J = (k_{33} k_{42} - k_{32} k_{43})(u^2 + v^2)(\lambda_1 \lambda_2 r_2 b + \lambda_1 \lambda_2 \overline{r_1^2 + r_2^2} b' + \lambda_1^2 r_1 a);$$

i.e.

$$J = \lambda_1 (k_{33} k_{42} - k_{32} k_{43})(u^2 + v^2)(\lambda_2 r_2 b + \lambda_1 r_1 a + \lambda_2 (r_1^2 + r_2^2) b') \quad . \quad (25)$$

From (19) we have,

$$k_{33} k_{42} - k_{32} k_{43} = \frac{1}{\Delta^2}[\{q_1 \lambda_2 - (q_1^2 + q_2^2) r_1 \lambda_1\}\{q_2 r_1 \lambda_1 \lambda_2 - q_1 r_2 \lambda_1^2\}$$
$$- \{q_1 r_1 \lambda_1^2 + (q_2 r_2 - 1)\lambda_1 \lambda_2\}\{(q_1^2 + q_2^2) r_2 \lambda_1 - q_2 \lambda_1\}].$$

The expression in the square brackets reduces to $q_2 \lambda_1 \Delta$, so that

$$k_{33} k_{42} - k_{32} k_{43} = \frac{q_2 \lambda_1}{\Delta}.$$

Hence

$$J = \frac{q_2 \lambda_1^2}{\Delta}(u^2 + v^2)(\lambda_2 r_2 b + \lambda_1 r_1 a + \lambda_2 \overline{r_1^2 + r_2^2} b').$$

We shall assume that m is not zero.

In this case it is easily seen that $\frac{q_2 \lambda_1^2}{\Delta}$ cannot vanish.

Furthermore, $u^2 + v^2$ cannot vanish unless u and v are each zero; *i.e.* unless $\theta = \frac{2\pi \lambda_2}{\lambda_1}$ is zero or a multiple of 2π.

But, since m is not zero, θ cannot vanish; so that u^2+v^2 cannot be zero unless λ_2 is a multiple of λ_1.

Again, using the relation $b' = -q_1 a - q_2 b$, we see that

$$\lambda_2 r_2 b + \lambda_1 r_1 a + \lambda_2(r_1^2+r_2^2)b' = a\{\lambda_1 r_1 - \lambda_2 q_1(r_1^2+r_2^2)\} + \lambda_2 b\{r_2 - q_2(r_1^2+r_2^2)\}.$$

In this expression the coefficient of a vanishes if $r_1^2+r_2^2 = \frac{\lambda_1}{\lambda_2}\frac{r_1}{q_1}$, and the coefficient of b vanishes if $r_1^2+r_2^2 = \frac{r_2}{q_2}$.

Now from (7)

$$r_1^2+r_2^2 = \frac{4n^2\lambda_2^2+p_{11}^2}{(\lambda_2^2+p_{20})}$$

and

$$\frac{\lambda_1}{\lambda_2}\frac{r_1}{q_1} = \frac{r_2}{q_2} = \frac{\lambda_1^2+p_{02}}{\lambda_2^2+p_{20}}.$$

Hence for either of these coefficients to vanish we must have

$$4n^2\lambda_2^2+p_{11}^2 = (\lambda_1^2+p_{02})(\lambda_2^2+p_{20});$$

i.e. from (5)

$$(\lambda_2^2+p_{20})(\lambda_1^2-\lambda_2^2) = 0.$$

Now $p_{20} = 3(1+\frac{1}{4}m)$ and is positive, so that $\lambda_2^2+p_{20}$ does not vanish. Hence J does not vanish unless λ_2 is equal to a multiple of λ_1. So that, provided λ_2/λ_1 is not an integer, there exist periodic solutions whose period is nearly equal to $\frac{2\pi}{\lambda_1}$.

Similarly, provided $\frac{\lambda_1}{\lambda_2}$ is not an integer, there exist periodic solutions whose period is nearly equal to $\frac{2\pi}{\lambda_2}$.

λ_1 and λ_2 are the roots of (5). Let λ_1 be the greater, so that

$$\lambda_1^2 = \frac{1+m+\sqrt{(1+m)^2-27m}}{2}$$

$$\lambda_2^2 = \frac{1+m-\sqrt{(1+m)^2-27m}}{2}.$$

We now see that for sufficiently small values of ϵ (*i.e.* for sufficiently small initial displacements from the equilibrium point B) periodic solutions exist with period nearly equal to $\frac{2\pi}{\lambda_1}$, and provided $\frac{\lambda_1}{\lambda_2}$ is not an integer, periodic solutions exist whose period is nearly equal to $\frac{2\pi}{\lambda_2}$.

In the case $\frac{\lambda_1}{\lambda_2}$ = an integer $\frac{2\pi}{\lambda_2}$ is a multiple of $\frac{2\pi}{\lambda_1}$, and the solutions with period nearly equal to $\frac{2\pi}{\lambda_1}$ may also be regarded as periodic solutions with period nearly equal to $\frac{2\pi}{\lambda_2}$.

But we have not been able (in the case when $\frac{\lambda_1}{\lambda_2}$ is an integer) to prove the existence of periodic solutions with period $\frac{2\pi}{\lambda_2}$ which are not at the same time periodic with period nearly equal to $\frac{2\pi}{\lambda_1}$.

Note.—The above work was completed in 1920 November, before the appearance of Moulton's *Periodic Orbits.* The details of the existence proofs are different from those of Buck, and it is hoped that they may be of interest. In Buck's paper, which apparently was completed in 1912 or earlier, the equations of motion are transformed and the jacobians take a relatively simple form. In this paper only two of the families of periodic orbits treated by Buck are discussed. A full account of the other families, and also of the actual development in series of the periodic solutions, is given in Buck's paper.

On Long-Period Inequalities in the Movements of Asteroids whose Mean Motions are nearly half that of Mars. By W. M. H. Greaves, B.A., Isaac Newton Student in the University of Cambridge.

(*Communicated by Professor H. F. Baker.*)

In the ordinary theory of the movements of the planets as developed by Laplace and Le Verrier, the equations of motion are integrated by a method of successive approximation with regard to the masses. The solutions in series thus obtained will be convergent for sufficiently small values of the time, but there is no guarantee that they will remain valid for all time. Generally speaking, the first two approximations will be sufficient, but it may happen that the mean motions of some of the bodies concerned are nearly commensurable, and in this case periodic terms with small denominators (and of long period) will appear in the first and succeeding approximations. The ordinary process would then be very tedious, as it would be necessary to proceed to a larger number of approximations.

The problem presented by asteroids whose mean motions are nearly double that of Jupiter has attracted a large amount of attention. It is sketched by Tisserand in t. iv. ch. xv. of his *Mécanique Céleste.* Tisserand reduces the disturbing function to the principal non-periodic and the principal long-period term, and endeavours to integrate the equations with the simplified disturbing function without appealing to successive approximations with respect to the mass of Jupiter.

In this paper we shall be concerned with asteroids whose mean motions are nearly half the mean motion of Mars. The existence of a group of such asteroids is shown by the following table :—

Asteroid.	Mean Daily Motion.	
No. 44 Nysa	941″·7363	
67 Asia	942·3560	
142 Polana	943·5246	Three values are taken from the *Connaissance de Temps*, 1915.
182 Elsa	944·5132	
265 Anna	942·0467	
622 Esther	944·890	

Mean motion of Mars = 1886·5183 (Le Verrier)

$= 2 \times 943''\cdot2592.$

At first sight one is tempted to cavil at Tisserand's procedure of integrating the equations of motion with the disturbing function reduced to a few terms. It is not my intention to dwell on this point, but it is perhaps worth while to point out that the method is formally equivalent to picking out various terms which would arise at different stages in the ordinary method with the *complete* disturbing function and grouping them together. For if we integrate by successive approximations with the reduced disturbing function, every term we obtain in the result will appear (among others) when we integrate by successive approximations with the complete disturbing function. And if when using the reduced disturbing function we do not integrate by successive approximations, but by some other method, we shall have obtained a way of grouping all these terms together.

The method of reducing the disturbing function to a few terms and then integrating was first used by Laplace in his discussion of the secular inequalities of the major planets. A sketch of this work, which was completed by Le Verrier, is given by Tisserand (*Mécanique Céleste*, t. i. ch. xxvi.).

The same method will be used in this paper, the object of which is to get a general idea of the nature and magnitude of the long-period inequalities due to the action of Mars in the movements of such asteroids as those mentioned above, and to compare them with the long-period inequalities due to the effect of Jupiter (the so-called "secular inequalities"). Precise results are not aimed at, and the problem is simplified at the outset by supposing the motion to be confined to one plane. For illustrative numerical purposes the four asteroids 44, 67, 142, and 182 will be taken. The inclinations of these are all comparatively small, and the results obtained should give a fairly good idea of the actual phenomena.

§ 1. *The Simplified Disturbing Function.*

We shall retain in the disturbing function the principal long-period terms due to the effect of Mars and the principal non-periodic terms due to the effect of Jupiter.

Let L = mean longitude of asteroid in the osculating ellipse.

L′ = mean longitude of Mars.

Let a, e, ϖ, ϵ denote the osculating elliptic elements of the asteroid at any time, a being the semi-major axis, e the eccentricity, ϖ the longitude of perihelion, and ϵ the mean longitude for the epoch.

Let a', e', ϖ', ϵ' denote the elements of Mars supposed constant.

Let a'', e'', ϖ'', ϵ'' denote the elements of Jupiter supposed constant.

Let m' denote the mass of Mars and m'' the mass of Jupiter, the mass of the sun being unity. We shall use astronomical units. Let κ^2 denote the constant of gravitation.

We then have

$$R = R_1 + R_2 + R_3 * \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R_1 = \frac{\kappa^2 m'}{2} e\left[3A^{(1)} + A_1^{(1)} - \frac{a}{a'^2}\right]\cos(2L - L' - \varpi)$$

$$R_2 = -\frac{\kappa^2 m'}{2} e\left[4A^{(2)} + A_1^{(2)}\right]\cos(2L - L' - \varpi')$$

$$\frac{R_3}{\kappa^2 m''} = \tfrac{1}{2}A^{(0)} + \tfrac{1}{4}(e^2 + e'^2)\{A_1^{(0)} + 2A_2^{(0)}\} + \tfrac{1}{2}\{A^{(1)} - A_1^{(1)} - 2A_2^{(1)}\}\cos(\varpi - \varpi'').$$

In the expressions for R_1 and R_2 the Le Verrier coefficients $A_s^{(i)}$ are to be calculated for the semi-major axes a and a' ($a' < a$), and in the expression for R_3 they are to be calculated for a and a'' ($a < a''$).

Transforming the expressions for R_1 and R_2 we obtain, using the usual notation, and putting $\frac{a'}{a} = \alpha$,†

$$R_1 = \frac{\kappa^2 m'}{2a}\left(3b^{(1)} + \alpha\frac{db^{(1)}}{d\alpha} - \frac{1}{\alpha^2}\right)e\cos(2L - L' - \varpi)$$

$$= \frac{\kappa^2 m'}{2a} eQ\cos(2L - L' - \varpi) \text{ say} \quad . \quad . \quad . \quad . \quad (2)$$

$$R_2 = -\frac{\kappa^2 m'}{2a}\left(4b^{(2)} + \alpha\frac{db^{(2)}}{d\alpha}\right)e'\cos(2L - L' - \varpi')$$

$$= -\frac{\kappa^2 m'}{2a} e'S\cos(2L - L' - \varpi') \text{ say} \quad . \quad . \quad . \quad (3)$$

Q and S are functions of α only.

Furthermore, on transforming the expression for R_3,‡

$$R_3 = \kappa^2 m'' M_0 + \kappa^2 m'' N_0(e^2 + e''^2) - 2\kappa^2 m'' P_0 e e''\cos(\varpi - \varpi'') \quad . \quad (4)$$

where

$$\left.\begin{aligned} M_0 &= \tfrac{1}{2}A^{(0)} \\ N_0 &= \tfrac{1}{8}B^{(1)} \\ P_0 &= \tfrac{1}{8}B^{(2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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* Tisserand, *Mécanique Céleste*, t. i. ch. xviii. pp. 309 and 311.

† *Ibid.*, t. i. ch. xvii. pp. 271 and 288.

‡ *Ibid.*, t. i. ch. xxvi. pp. 405 and 406.

We shall reduce the disturbing function to R_1, R_2, and R_3 successively, and integrate the equations of motion in each case, thus obtaining three inequalities of long period, the first depending on the eccentricity of the asteroid, the second depending on the eccentricity of Mars, and the third arising out of the secular action of Jupiter.

§ 2. *Inequalities depending on the Eccentricity of the Asteroid.*

We reduce the disturbing function to the portion R_1.
Write

$$L = \kappa\sqrt{a}$$
$$G = \kappa\sqrt{a(1-e^2)}.$$

l = mean anomaly in osculating ellipse.
The equations of motion of the asteroid are

$$\frac{dL}{dt} = \frac{\partial F'}{\partial l} \qquad \frac{dl}{dt} = -\frac{\partial F'}{\partial L}$$
$$\frac{dG}{dt} = \frac{\partial F'}{\partial \varpi} \qquad \frac{\partial \varpi}{\partial t} = -\frac{\partial F'}{\partial G}$$

where

$$F' = \frac{\kappa^2}{2a} + R_1.$$

Putting $g = \varpi - n't - \epsilon' = \varpi - L'$, n' being the mean motion of Mars, these equations become,

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l} & \frac{dl}{dt} &= -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g} & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

where

$$F = F' + n'G = \frac{\kappa^2}{2a} + n'G + \frac{\kappa^2 m'}{2a} eQ\cos(2l+g) \quad . \quad . \quad (7)$$

We also have the relations

$$n'^2 a'^3 = \kappa^2(1+m') \quad . \quad . \quad . \quad . \quad . \quad (7)$$
$$n^2 a^3 = \kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where n is the "mean motion" of the asteroid.
We write

$$F = A\cos\theta + B$$
$$A = \frac{\kappa^2 m'}{2a} eQ$$
$$B = \frac{\kappa_2}{2a} + n'\kappa\sqrt{a(1-e^2)}$$
$$\theta = 2l + g.$$

Then from (6)

$$\frac{dL}{dt} = -2A\sin\theta \qquad \frac{dG}{dt} = -A\sin\theta.$$

$$\therefore\ L - 2G = \text{a constant} \quad . \quad . \quad . \quad . \quad (9)$$

We also have the integral

$$F = A\cos\theta + B = C. \quad . \quad . \quad . \quad . \quad (10)$$

where C is a constant.

We now get

$$\left(\frac{dL}{dt}\right)^2 = 4A^2\sin^2\theta = 4\left(A^2 - \overline{C-B}^2\right).$$

$$\therefore\ 2dt = \frac{\pm\, dL}{\sqrt{A^2 - (C-B)^2}}. \quad . \quad . \quad . \quad . \quad (11)$$

A and B are given as functions of a and e and are, therefore, functions of L and G. Hence by using (9) it is possible to find them as functions of L only, and the problem is reduced to the inversion of the integral (11).

To do this, let a_1, n_1, e_1, ϖ_1, l_1, g_1, θ_1, Q_1 be the initial values of a, n, e, ϖ, l, g, θ, and Q.

Write

$$a = a_1(1+x).$$

e can be expanded in powers of x by using (9). We find

$$e^2 = {e_1}^2 + x\sqrt{1-{e_1}^2}\{\sqrt{1-{e_1}^2} - \tfrac{1}{2}\} + o(x^2).$$

Expanding in powers of x we obtain,

$$A = \frac{\kappa^2 m'}{2a_1}[e_1Q_1 + E_1x + o(x^2)] \quad . \quad . \quad . \quad . \quad (12)$$

$$C - B = \frac{\kappa^2}{2a_1}[m'e_1Q_1\cos\theta_1 - (D_1x + D_2x^2 + o(x^3))] \quad . \quad (13)$$

where

$$E_1 = -e_1Q_1 + Q_1\sqrt{1-{e_1}^2}\ \ \frac{\sqrt{1-{e_1}^2} - \frac{1}{2}}{2e_1} - e_1a_1\left(\frac{\partial Q}{\partial a}\right)_1^*$$

$$D_1 = \tfrac{1}{2}\frac{n'}{n_1} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$D_2 = 1 - \tfrac{1}{8}\frac{n'}{n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Hence

$$A^2 - (C-B)^2 = \frac{\kappa^4}{4{a_1}^4}[m'^2\{e^2{Q_1}^2 + 2e_1Q_1E_1x + \ldots\} - \{m'e_1Q_1\cos\theta_1 - (D_1x + D_2x^2 + \ldots)\}^2]$$

$$= \frac{\kappa^4}{4{a_1}^4}U \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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* $\left(\frac{\partial Q}{\partial a}\right)_1$ denotes the initial value of $\frac{\partial Q}{\partial a}$.

Hence from (1), using $L \mp \kappa\sqrt{a} = \kappa\sqrt{a_1(1+x)}$, we obtain

$$2n_1 dt = \frac{\pm dx}{\sqrt{u(1+x)}} \quad . \quad . \quad . \quad . \quad (17)$$

We can write

$$U = U_1 U_2 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where

$$\left.\begin{aligned} U_1 &= m'\{2e_1Q_1 \sin^2 \tfrac{1}{2}\theta_1 + E_1 x + \ldots\} + (D_1 x + D_2 x^2 + \ldots) \\ U_2 &= m'\{2e_1Q_1 \cos^2 \tfrac{1}{2}\theta_1 + E_1 x + \ldots\} - (D_1 x + D_2 x^2 + \ldots) \end{aligned}\right\} \quad (19)$$

We then find

$$A + C - B = 2A \cos^2 \frac{\theta}{2} = \frac{\kappa^2}{2a_1} U_2$$

$$A - C + B = 2A \sin^2 \frac{\theta}{2} = \frac{\kappa^2}{2a_1} U_1,$$

so that

$$\cos\theta = \frac{m'e_1Q_1 \cos\theta_1 - D_1 x - D_2 x^2 - \ldots}{m'(e_1Q_1 + E_1 x + \ldots)} \quad . \quad . \quad (20)$$

For the asteroids with which we are dealing D_1 will be small. We shall neglect terms of order x^3, terms of order $m'x$, and terms of order $D_1 x^2$.

We now have approximately

$$U_1 = 2m'e_1Q_1 \sin^2 \tfrac{1}{2}\theta_1 + D_1 x + D_2 x^2$$
$$U_2 = 2m'e_1Q_1 \cos^2 \tfrac{1}{2}\theta_1 - D_1 x - D_2 x^2,$$

giving

$$2n_1 dt = \frac{\pm dx}{\sqrt{\left\{2m'e_1Q_1 \sin^2 \frac{\theta_1}{2} + D_1 x + D_2 x^2\right\}\left\{2m'e_1Q_1 \cos^2 \frac{\theta_1}{2} - D_1 x - D_2 x^2\right\}}}.$$

Writing $x = y - \dfrac{D_1}{2D_2}$, this gives

$$2n_1 D_2 dt = \frac{\pm dy}{\sqrt{\left\{y^2 + \frac{8m'D_2e_1Q_1 \sin^2 \frac{1}{2}\theta_1 - D_1^2}{4D_2^2}\right\}\left\{\frac{8m'D_2e_1Q_1 \cos^2 \frac{1}{2}\theta_1 + D_1^2}{4D_2^2} - y^2\right\}}} \quad . \quad (21)$$

Notice that D_2 is positive.

We now have two cases to consider.

Case A :—Suppose

$$D_1^2 > 8m'D_2e_1Q_1 \sin^2 \tfrac{1}{2}\theta_1.$$

Write

$$p^2 = \frac{D_1^2 - 8m'D_2e_1Q_1 \sin^2 \frac{1}{2}\theta_1}{4D_2^2} \quad . \quad . \quad . \quad (22)$$

$$p'^2 = \frac{D_1^2 - 8m'D_2e_1Q_1 \cos^2 \frac{1}{2}\theta_1}{4D_2^2} \quad . \quad . \quad . \quad (23)$$

(21) becomes

$$2n_1 D_2 dt = \frac{\pm dy}{\sqrt{(y^2 - p^2)(p'^2 - y^2)}}$$

This gives on integration

$$y = \pm p' dnu \quad . \quad . \quad . \quad . \quad (24)$$

where

$$u = 2p' n_1 D_2 (t + c) \quad . \quad . \quad . \quad . \quad (25)$$

c being a constant.

And the elliptic function dnu is to be formed with modulus k given by

$$k^2 = 1 - \frac{p^2}{p'^2} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Hence

$$x = \pm p' dn\{2p' n_1 D_2 (t + c)\} - \frac{D_1}{2D_2} \quad . \quad . \quad . \quad (27)$$

a is then given by $a = a_1(1 + x)$.

The ambiguity of sign in (27) must be settled by the consideration that $x = 0$ initially. If D_1 is positive we take the + sign, and if D_1 is negative we take the − sign.

The range of variation of x is denoted by R_x, and is given by

$$R_x = p'(1 - \sqrt{1 - k^2}) = p' - p,$$

and if R_a is the range of variation of a we get

$$R_a = a_1(p' - p) \quad . \quad . \quad . \quad . \quad (28)$$

With our methods of approximation (20) becomes

$$\cos\theta = \frac{m' e_1 Q_1 \cos\theta_1 - D_1 x - D_2 x^2}{m' e_1 Q_1}.$$

Substituting for x in terms of y and simplifying, we get

$$\cos\theta = 2sn^2 u - 1,$$

so that, if

$$\Phi = amu \qquad \cos(\pi - \theta) = \cos 2\Phi.$$

$$\therefore \ \pi - \theta = \pm 2\Phi$$

$$= \pm 2am(2p' n_1 D_2 (t + c)) \quad . \quad . \quad . \quad . \quad (29)$$

Notice that θ can take all values.

x is a periodic function of t. The period T is given by

$$T = \frac{K}{p' n_1 D_2} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where κ is the complete elliptic integral

$$K = \int_0^{\frac{\pi}{2}} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}}.$$

The constant c can be determined from the fact that, when $t=0$, $x=0$; or from the fact that when $t=0$, $\theta=\theta_1$.

Case B :—

$$D_1^2 < 8m'D_2e_1Q_1\sin^2\tfrac{1}{2}\theta_1.$$

Put

$$p''^2 = \frac{8m'D_2e_1Q_1\sin^2\frac{1}{2}\theta_1 - D_1^2}{4D_2^2} \quad . \quad . \quad . \quad (31)$$

We get

$$2n_1D_2dt = \frac{\pm dy}{\sqrt{(y^2+p''^2)(p^2-y^2)}}$$

p' being given by (23).

Leading to

$$x = \pm p'cnu - \frac{D_1}{2D_2} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$u = 2n_1D_2\sqrt{(p'^2+p''^2)}(t+c) \quad . \quad . \quad . \quad (33)$$

k, the modulus of the elliptic function, is given by

$$k^2 = \frac{p'^2}{p'^2+p''^2} \quad . \quad . \quad . \quad . \quad (34)$$

In this case we find

$$\cos\theta = 1 - 2dn^2u.$$

Putting

$$dnu = \cos\chi \quad . \quad . \quad . \quad . \quad . \quad (35)$$

we have

$$\theta = 180^\circ \pm 2\chi \quad . \quad . \quad . \quad . \quad (36)$$

In this case θ cannot take all values, but librates between the values $180^\circ \pm \chi'$, where

$$\chi' = \sin^{-1}k.$$

x and θ are periodic functions of period T, where

$$T = \frac{2K}{n_1D_2\sqrt{p'^2+p''^2}} \quad . \quad . \quad . \quad . \quad (37)$$

also R_a = range of variation of $a = 2ap'$ . . . . . (38)

The case in which $D_1^2 = 8m'D_2e_1Q_1\sin^2\frac{1}{2}\theta_1$ can be deduced from either of the cases A or B by putting $k=1$. In this case we have

$$dnu = cnu = \operatorname{sech} u$$

$$\Phi = amu = gdu \qquad u = \log_e\tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right)$$

$$u = 2p'n_1D_2(t+c)$$

$$x = \pm p'\operatorname{sech} u - \frac{D_1}{2D_2}$$

$$\theta = 180^\circ \pm 2\Phi.$$

Inequalities in Mean Longitude.

We have

$$L = \int n dt + \epsilon,$$

so that

$$\frac{dL}{dt} = n + \frac{d\epsilon}{dt} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\frac{\partial R_1}{\partial a} + \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R_1}{\partial e},$$

R_1 being given by equation (2).

Here in forming the derivative $\frac{\partial R_1}{\partial a}$ we only make a vary in the factor $\frac{\kappa^2 m'}{2a} e Q$ of R_1, and not in the argument $2L - L' - \varpi$, where a is involved through the quantity n occurring in L.*

Performing the differentiations, expanding in powers of x and neglecting terms of order $m'x$, we obtain

$$\frac{d\epsilon}{dt} = n_1(\mu + \nu)\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where

$$\mu = m' e_1 \left(Q_1 + a_1 \left(\frac{\partial Q}{\partial a}\right)_1\right) \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\nu = \tfrac{1}{2} m' Q_1 \sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{e_1}\cos\theta \quad . \quad . \quad . \quad (42)$$

From 8 we deduce

$$n = n_1(1+x)^{-\frac{3}{2}}$$
$$= n_1\left(1 - \tfrac{3}{2}x + \frac{15}{8}x^2 + \ldots\right) \quad . \quad . \quad . \quad . \quad (43)$$

L is to be obtained from (39), (40), (43), and the expressions for x and θ in terms of the time.

We have to separate the two cases A and B as before.

Case A.—We have from (30)

$$\int \cos\theta dt = -\frac{1}{2p' n_1 D_2}\int \cos 2\Phi du$$

and

$$\int \cos 2\Phi du = u - 2\int sn^2 u du$$
$$= -\left(\frac{2}{k^2} - 1\right)u + \frac{2}{k^2}E(u),$$

where, as usual,

$$E(u) = \int_0^u dn^2 u du.$$

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* Tisserand, *Mécanique Céleste*, t. i. ch. xi. pp. 191 to 193.

We introduce the function $Z(u)$ defined by the equation

$$E(u) = Z(u) + \frac{E}{K}u,$$

where E and K are the complete elliptic integrals, $Z(u)$ being a periodic function of u with period $2K$.

We then obtain

$$\int \cos\theta dt = \left(\frac{2}{k^2}\frac{K-E}{K} - 1\right)t - \frac{1}{k^2 p' n_1 D_2} Z(u) \quad . \quad . \quad (44)$$

from (40)

$$\epsilon = n_1(\mu+\nu)\left[\left\{\frac{2}{k^2}\frac{K-E}{K} - 1\right\}t - \frac{1}{k^2 p' n_1 D_2} Z(u)\right] + \text{a constant} \quad (45)$$

Again,

$$\int dnudu = \Phi \quad \text{and} \quad \int dn^2 u du = E(u).$$

We obtain from (43) after integration

$$\int ndt = n_1 t - \tfrac{3}{2}n_1\left[\frac{\pm\Phi}{2n_1 D_2} - \frac{D_1 t}{2D_2}\right] + \frac{15}{8}n_1\left[\frac{p'}{2n_1 D_2}Z(u) \mp \frac{\Phi D_1}{2n_1 D_2^2} + t\left\{p'^2\frac{E}{K} + \frac{D_1^2}{4D_2^2}\right\}\right] \quad . \quad (46)$$

We can now obtain L from (39), using (45) and (46).

Writing $L = n_1 t + \epsilon_1 + \delta L$ we have, on adjusting the constants so that $\delta L = 0$ when $t = 0$,

$$\begin{aligned}\delta L = {} & \frac{1}{D_2}\left[\frac{15}{16}p' - \frac{\mu+\nu}{k^2 p'}\right]\{Z(u) - Z(u_1)\} \\ & \mp \frac{3}{4D_2}(\Phi - \Phi_1) \mp \frac{15 D_1}{16 D_2^2}(\Phi - \Phi_1) \\ & + \tfrac{3}{4}\frac{n_1 D_1}{D_2}t + \frac{15}{8}n_1\left\{p'^2\frac{E}{K} + \frac{D_1^2}{4D_2^2}\right\}t \\ & + (\mu+\nu)n_1\left\{\frac{2}{k^2}\frac{K-E}{K} - 1\right\}t \quad . \quad . \quad . \quad . \quad (47)\end{aligned}$$

where Φ_1 and u_1 are the initial values of Φ and u.

Introduce the function $\lambda(u)$ defined by

$$\lambda(u) = \Phi - \frac{\pi u}{2K} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$\lambda(u)$ is then a periodic function of u with period $2K$.

We then obtain from (47)

$$\left.\begin{aligned}\delta L = {} & \mp \frac{3}{4D_2}\{\lambda(u) - \lambda(u_1)\} \\ & \mp \frac{15D_1}{16D_2^2}\{\lambda(u) - \lambda(u_1)\} + \frac{1}{D_2}\left\{\frac{15}{16}p' - \frac{\mu+\nu}{k^2 p'}\right\}\{Z(u) - Z(u_1)\}\end{aligned}\right\} \text{Periodic terms.}$$

$$\left.\begin{aligned} & \mp \frac{3\pi}{4K}p' n_1 t \mp \frac{15D_1}{16D_2}\frac{\pi}{K}p' n_1 t \\ & + \tfrac{3}{4}\frac{n_1 D_1}{D_2}t + \frac{15}{8}n_1\left\{p'^2\frac{E}{K} + \frac{D_1^2}{4D_2^2}\right\}t \\ & + (\mu+\nu)n_1\left\{\frac{2}{k^2}\frac{K-E}{K} - 1\right\}t\end{aligned}\right\} \text{Secular terms.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

If we approximate still further and only retain the principal periodic term and the principal secular term, we get

$$\delta L = \mp \frac{3}{4D_2}\{\lambda(u) - \lambda(u_1)\} + \tfrac{3}{4}n_1 t\left\{\frac{D_1}{D_2} \mp \frac{\pi}{K}p'\right\} \quad . \quad . \quad (49')$$

Let A denote the amplitude (*i.e.* half the total range of variation) of the principal periodic term $\mp \frac{3}{4D_2}\lambda(u)$ in δL.

Then it is easily found that (using the relation $\frac{d}{du}amu = dnu$)

$$A = \lambda(u') \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where u' is given by

$$dnu' = \frac{\pi}{2K} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Case B.—In this case we obtain, using a similar method of procedure,*

$$\left.\begin{aligned}\delta L = {} & \mp \frac{3}{4D_2}(\chi - \chi_1) + \frac{15}{16}\frac{p'}{D_2 k}[Z(u) - Z(u_1)] \\ & \mp \frac{15}{16}\frac{D_1}{D_2^2}(\chi - \chi_1) - \frac{k}{p' D_2}(\mu+\nu)[Z(u) - Z(u_1)]\end{aligned}\right\} \text{Periodic terms.}$$

$$\left.\begin{aligned} & + \tfrac{3}{4}\frac{D_1}{D_2}n_1 t + \frac{15}{8}p'^2\left[1 - \frac{1}{k^2}\frac{K-E}{K}\right]n_1 t \\ & + \frac{15}{32}\frac{D_1^2}{D_2^2}n_1 t + (\mu+\nu)\left[1 - \frac{2E}{K}\right]n_1 t\end{aligned}\right\} \text{Secular terms.} \quad . \quad . \quad . \quad (52)$$

where u and χ are given by equations (33) and (35) and u_1 and χ_1 are their initial values.

* Note that $\int cnu\,du = \frac{\chi}{k}$ where $dnu = \cos\chi$, as in (35).

Note that the ambiguities of sign in (49) and (52) correspond to the ambiguities in (27) and (32), which are to be settled by the sign of D_1. If D_1 is positive we take the upper sign, and if D_1 is negative we take the lower.

Retaining the principal terms in (52), we obtain

$$\delta L = \mp \frac{3}{4D_2}(\chi - \chi_1) + \tfrac{3}{4}\frac{D_1}{D_2} n_1 t \quad . \quad . \quad . \quad (52')$$

We also have

$$A = \frac{3}{4D_2} \sin^{-1} k . \quad . \quad . \quad . \quad . \quad (53)$$

where as before A is the amplitude of the principal periodic term in δL.

Note that in the limiting case, when $k = 1$,

$$A = \frac{3}{4D_2} \times 90^\circ = 90^\circ \text{ approx.}$$

Inequalities in Eccentricity.

On page 153 we obtained the relation

$$e^2 = e_1^2 + x\sqrt{1-e_1^2}\{\sqrt{1-e_1^2} - \tfrac{1}{2}\} + o(x^2).$$

From this e can be obtained from (27) for Case A, and from (32) for Case B.

Let R_{e^2} denote the range of variation of e^2. It is easily seen that for Case A

$$R_{e^2} = \sqrt{1-e_1^2}\{\sqrt{1-e_1^2} - \tfrac{1}{2}\}(p' - p) \quad . \quad . \quad (54)$$

For Case B

$$R_{e^2} = 2p'\sqrt{1-e_1^2}\{\sqrt{1-e_1^2} - \tfrac{1}{2}\} \quad . \quad . \quad . \quad (55)$$

§ 3. *Inequalities depending on the Eccentricity of Mars.*

We reduce the disturbing function to the portion R_2.

The equations of motion are as before, R_2 being substituted for R_1.

We write

$$g = \varpi - \tfrac{1}{2}(n't + \epsilon' - \varpi')$$

and the equations become

$$\frac{dL}{dt} = \frac{\partial F}{\partial l} \quad \frac{dG}{dt} = \frac{\partial f}{\partial g} \quad \frac{dl}{dt} = -\frac{\partial F}{\partial L} \quad \frac{\partial g}{dt} = -\frac{\partial F}{\partial G},$$

where

$$F = \frac{\kappa^2}{2a} + \tfrac{1}{2}n'\kappa\sqrt{a(1-e^2)} - \frac{\kappa^2 m'}{2a} e' S \cos 2(l+g).$$

We write $l + g = \theta$, $F = -A\cos 2\theta + B = C$, where C is a constant.

Proceeding as before, we have

$$L - G = \text{a constant}$$

$$2dt = \frac{\pm dL}{\sqrt{A^2 - (B-C)^2}}.$$

Note that in this case we also have

$$g = \tfrac{1}{2}n't + \text{a constant},$$

so that

$$\varpi = \text{a constant}.$$

As in § 2, we put $a = a_1(1+x)$ and expand in powers of x. We obtain, the procedure being the same as in the last section,

$$e^2 = e_1^2 - x\sqrt{1-e_1^2}(1-\sqrt{1-e_1^2})$$

$$\cos 2\theta = \frac{m'e'S_1\cos^2\theta_1 + D_1x + D_2x^2}{e'm'S_1},$$

D_1 and D_2 being given by (14) and (15).

$$2n_1D_2dt = \frac{\pm dy}{\sqrt{\left(y^2 + \frac{8m'D_2e'S_1\cos^2\theta_1 - D_1^2}{4D_2^2}\right)\left(\frac{8m'D_2e'S_1\sin^2\theta_1 + D_1^2}{4D_2^2} - y^2\right)}}$$

where

$$x = y - \frac{D_1}{2D_2}.$$

As before, we have two cases to consider, and we get the following results. The method of procedure is the same.

Case A:—

$$D_1^2 > 8m'D_2e'S_1\cos^2\theta_1.$$

We put

$$\frac{D_1^2 - 8m'D_2e'S_1\cos^2\theta_1}{4D_2^2} = q^2 \qquad \frac{D_1^2 + 8m'D_2e'S_1\sin^2\theta_1}{4D_2^2} = q'^2 \quad . \quad (56)$$

$$k^2 = 1 - \frac{q^2}{q'^2} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$u = 2q'n_1D_2(t+C) \quad . \quad . \quad . \quad . \quad (58)$$

and we get

$$x = \pm q'dnu - \frac{D_1}{2D_2} \quad . \quad . \quad . \quad . \quad (59)$$

where the upper sign is to be taken if D_1 is positive, and the lower if D_1 is negative.

Also

$$\theta = r\pi \pm \Phi,$$

where r is any integer. Notice that θ can take all values.

We also get for the perturbation in longitude δL,

$$\begin{aligned}\delta L = &\mp \frac{3}{4D_2}\frac{D_1}{D_2^2}(\lambda(u)-\lambda(u_1)) + \frac{15}{16}\frac{q'}{D_2}(Z(u)-Z(u_1))\\ &-\frac{\mu' e'}{k^2 q' D_2}(Z(u)-Z(u_1))\\ &+\left(\frac{3D_1}{4D_2}\mp\frac{3\pi}{4K}q'\right)n_1 t\\ &+\left[\frac{15q'^2}{8}\frac{E}{K}+\frac{15}{32}\frac{D_1^2}{D_2^2}\mp\frac{15\pi}{16K}q'\frac{D_1}{D_2}\right]n_1 t \quad . \quad . \quad . \quad (60)\end{aligned}$$

where $\lambda(u)$ is defined as before and μ' is given by

$$\mu' = m'e'\left(S + a_1\left(\frac{\partial S}{\partial a}\right)_1\right).$$

Retaining only the principal terms, we have

$$\delta L = \mp\frac{3}{4D_2}[\lambda(u)-\lambda(u_1)] + \frac{3}{4}\left[\frac{D_1}{D_2}\mp\frac{\pi}{K}q'\right]n_1 t \quad . \quad . \quad (60')$$

the upper or lower sign being taken according as D_1 is positive or negative. We also have

A = amplitude of principal periodic inequality in δL

$$= \frac{3}{4D_2}\lambda(u') \quad \text{where} \quad dnu' = \frac{\pi}{2K}$$

R_a = range of variation of $a = a_1(q'-q)$

R_{e^2} = range of variation of $e^2 = \sqrt{1-e_1^2}\{1-\sqrt{1-e_1^2}\}(q'-q)$

$$T = \text{period} = \frac{K}{q' n_1 D_2}.$$

Case B :—

$$D_1^2 < 8m'D_2 e' S_1 \cos^2\theta_1.$$

We put

$$\left.\begin{aligned}q'^2 &= \frac{8m'D_2e'S_1\sin^2\theta_1 + D_1^2}{4D_2^2}\\ q''^2 &= \frac{8m'D_2e'S_1\cos^2\theta_1 - D_1^2}{4D_2^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (61)$$

$$k^2 = \frac{q'^2}{q'^2+q''^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$u = 2n_1 D_2\sqrt{q'^2+q''^2}(t+c),$$

and we get

$$x = \pm q' cnu - \frac{D_1}{2D_2}$$

$$\theta = r\pi \pm \chi \quad \text{where} \quad dnu = \cos\chi.$$

Amplitude of libration of $\theta = \sin^{-1} k$.

$$\begin{aligned}\delta L = &\mp \frac{3}{4D_2}(\chi - \chi_1) + \frac{15}{16}\frac{q'}{kD_2}[Z(u) - Z(u_1)] \mp \frac{15}{16}\frac{D_1}{D_2}(\chi - \chi_1) \\ &- \frac{k\mu' e'}{q' D_2}[Z(u) - Z(u_1)] \\ &+ \frac{3D_1}{4D_2} n_1 t + \frac{15 q'^2}{8}\left\{1 - \frac{1}{k^2}\frac{K - E}{K}\right\} n_1 t + \frac{15}{32}\frac{D_1^2}{D_2^2} n_1 t \\ &- \mu' e'\left[2\frac{E}{K} - 1\right] n_1 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)\end{aligned}$$

Retaining only the principal terms, we get

$$\delta L = \mp \frac{3}{4D_2}(\chi - \chi_1) + \frac{3D_1}{4D_2} n_1 t \quad . \quad . \quad . \quad . \quad . \quad . \quad (63')$$

A = amplitude of principal periodic term in $\delta L = \mp \frac{3}{4D_2} \sin^{-1} k$

$R_a = 2a_1 q'$

$R_{e^2} = 2q'\sqrt{1 - e_1^2}\{1 - \sqrt{1 - e_1^2}\}$

$$T = \frac{2K}{n_1 D_2 \sqrt{q'^2 + q''^2}}.$$

The ambiguities of sign in the expressions for x and δL are to be settled by the rule already given.

§ 4. *Secular Inequalities arising from the Action of Jupiter.*

We reduce the disturbing function to the portion R_3.

We have at once

$$\begin{aligned} a &= \text{const.} \\ n &= \text{const.} \end{aligned}$$

so that there are no long-period inequalities in the major axis arising from this cause.

Put

$$\left.\begin{aligned} e \sin \varpi &= h \\ e \cos \varpi &= l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The equations of motion become approximately *

$$\left.\begin{aligned} \frac{dh}{dt} &= \frac{1}{na^2}\frac{\partial R_3}{\partial l} \\ \frac{dl}{dt} &= -\frac{1}{na^2}\frac{\partial R_3}{\partial h} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (65)$$

* Tisserand, *Mécanique Céleste*, t. i. p. 407.

We easily find

$$\left.\begin{aligned} \frac{1}{na^2}\frac{\partial R_3}{\partial l} &= \beta l - \gamma l' \\ \frac{1}{na^2}\frac{\partial R_3}{\partial h} &= \beta h - \gamma h' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (66)$$

where

$$\left.\begin{aligned} \beta &= \frac{2\kappa^2 m'}{na^2} N_0 \\ \gamma &= \frac{2\kappa^2 m'}{na^2} P_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (67)$$

so that we obtain the equations

$$\left.\begin{aligned} \frac{dh}{dt} &= \beta l - \gamma l' \\ \frac{dl}{dt} &= -\beta h + \gamma h' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (68)$$

giving

$$\left.\begin{aligned} h &= \bar{A}\sin(\beta t + \lambda) + \frac{\gamma}{\beta} e'' \sin \varpi'' \\ l &= \bar{A}\cos(\beta t + \lambda) + \frac{\gamma}{\beta} e'' \cos \varpi'' \end{aligned}\right\} \quad . \quad . \quad . \quad (69)$$

where $\bar{A}$ and λ are constants.

Putting $t = 0$ in (69) we obtain

$$\bar{A}^2 = e_1^2 + \frac{\gamma^2}{\beta^2} e''^2 - \frac{2\gamma}{\beta} e_1 e'' \cos(\varpi_1 - \varpi'') \quad . \quad . \quad (70)$$

also $e^2 = h^2 + l^2$; and from (69) we easily find

$$R_e^2 = \text{range of variation of } e^2 = 2Ae''\frac{\gamma}{\beta} \quad . \quad . \quad (71)$$

For ϵ we have the approximate equation *

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\frac{\partial R_3}{\partial a} + \frac{e}{2na^2}\frac{\partial R_3}{\partial e} \quad . \quad . \quad . \quad (72)$$

e is now known from (69), so that (72) can be integrated.

The result is

$$\epsilon = \epsilon_0 + Ht + A\sin(\beta t + \lambda - \varpi'') \quad . \quad . \quad . \quad (73)$$

where

$$H = \frac{\kappa^2 m''}{na^2}\left[\left\{N_0 - 2a\frac{dN_0}{da}\right\}\left\{\bar{A}^2 + \frac{\gamma^2}{\beta^2}e''^2\right\} - \left\{P_0 - 4a\frac{dP_0}{da}\right\}\frac{\gamma}{\beta}e''^2 - 2a\frac{dM_0}{da} - 2e''^2 a\frac{dN_0}{da}\right]$$

* Tisserand, *Mécanique Céleste*, t. i. ch. xxvi. p. 428.

and

$$A = \frac{\kappa^2 m''}{na^2}\frac{\bar{A}e''}{\beta^2}\{2\gamma N_0 - \beta P_0\} + \frac{\kappa^2 m''}{na^2}\frac{\bar{A}e''}{\beta^2}\left\{4\beta a\frac{dP_0}{da} - 4\gamma a\frac{dN_0}{da}\right\},$$

ϵ_0 being a constant.

Now if L be the mean longitude $L = nt + \epsilon = n_1 t + \epsilon_1 + \delta L$ say.

So that $\delta L = \epsilon - \epsilon_1 + (n - n_1)t$, ϵ_1 being the initial value of ϵ.

So that A is the amplitude of the principal periodic term in δL.

We find on reduction, substituting for γ, β, P_0, and N_0 from (5) and (67),

$$A = \tfrac{1}{2}\bar{A}e''\frac{B^{(2)}}{B^{(1)}} + 2\bar{A}e'a\frac{d}{da}\left(\frac{B^{(2)}}{B^{(1)}}\right) \quad . \quad . \quad . \quad (74)$$

also, if T be the period,

$$T = \frac{2\pi}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (75)$$

§ 5. *Numerical Applications.*

For illustrative numerical purposes the asteroids 44, 67, 142, and 182 were selected.

The following data are taken from the supplement to the *Connaissance des Temps*, 1915:—

Asteroid.	Epoch of Osculation.	Log a.	M.	ϕ.
44 Nysa	1891 Apr. 1·0	0·3840515	101° 30′ 7″·1	8° 48′ 10″·9
67 Asia	1897 Dec. 5·0	0·3838611	201 21 25·2	10 47 54·5
142 Polana	1896 Dec. 10·0	0·3835023	211 13 22·8	7 44 10·6
182 Elsa	1897 Mar. 20·0	0·3831990	102 52 20·2	10 50 51·9

	i.	Ω.	ω.
44 Nysa	3° 41′ 57″·2	131° 30′ 16″·7	340° 33′ 54″·7
67 Asia	5 29 6·4	203 12 54·9	103 19 53·9
142 Polana	2 14 31·3	292 11 48·7	289 56 53·7
182 Elsa	2 10 7·3	106 53 6·8	308 18 36·2

a = semi-axis major. Ω = longitude of ascending node.
$e = \sin\phi$ = eccentricity. ω = angular distance from node to perihelion.
i = inclination. M = mean anomaly at epoch of osculation.
i, ω, and Ω are referred to mean ecliptic and equinox for 1920·0.

The inclinations will be neglected and the preceding work applied, the long-period inequalities of the three kinds discussed being investigated.

It is found that the asteroid 142 Polana falls within the Case B (the case of libration) as regards both of the first two inequalities. The other three asteroids fall under Case A as regards each of the first two inequalities.

In applying the theory of the previous sections, the epoch of oscula-

tion was taken as the origin of time for each asteroid and the quantities T, A, R_a, and R_{e^2} computed for each of the three inequalities considered. In the computation the values of the Le Verrier coefficients were derived by interpolation from Runkle's tables. The mean longitudes of Mars at the origin of time for each asteroid were derived from the tables of Le Verrier and Gaillot, as quoted in the *Connaissance des Temps*. The values of the elements a, e, ϖ for Jupiter and Mars have been assumed to be constant in the theory. The following values were derived from the tables in the *Connaissance des Temps*, and adopted for the purposes of the computation.

For Mars—

$\log a' = 0{\cdot}1828932.$
$e' = 0{\cdot}0933088$, being the value for 1900 Jan. 1·0.
$\varpi' = 334^\circ\ 29'\ 51''$, being the value for 1900 Jan. 1·0 reduced to mean ecliptic and equinox for 1920 Jan. 1·0.

For Jupiter—

$\log a'' = 0{\cdot}716217.$
$e'' = 0{\cdot}04832654$, being the value for 1895 Jan. 1·0.
$\varpi'' = 12^\circ\ 59'\ 22''{\cdot}10$, being the value for 1895 Jan. 1·0 reduced to mean ecliptic and equinox for 1920 Jan. 1·0.

The following values of the masses were used:—

For Mars, $m' = \frac{1}{3093500}$. For Jupiter, $m'' = \frac{1}{1047{\cdot}355}$.

The results are as follows:—

Long-period Inequality I. (due to the action of Mars, and arising from a term of long period in the disturbing function involving the eccentricity of the asteroid).

Asteroid.	T (in Julian Years).	A.	R_a (in Astronomical Units).	R_{e^2}.
44	1162·9	0° 41′	0·000125	0·00002488
67	1950·4	2 22	0·000257	0·00005037
182	1483·6	1 25	0·000201	0·00003941
142	9030·7	39 57	0·001960	0·00019722

Long-period Inequality II. (due to the action of Mars, and arising from a term of long period in the disturbing function involving the eccentricity of Mars).

Asteroid.	T (in Julian Years).	A.	R_a (in Astronomical Units).	R_{e^2}.
44	1198·4	1° 13′	0·000124	0·00000057
67	2070·6	3 42	0·000380	0·00000273
182	1368·1	1 38	0·000254	0·00000184
142	8065·7	61 58	0·003734	0·00000696

Long-period Inequality III. (due to the action of Jupiter).

Asteroid.	T (in Julian Years).	A.	R_a.	R_{e^2}.
44	32938	35′ 30″	0	0·00871460
67	32985	31 29	0	0·00772540
182	33119	30 51	0	0·00920888
142	33057	37 34	0	0·00757000

In these tables—

T = period of inequality.

A = amplitude (*i.e.* half the total range of variation) of the principal periodic term in the expression for the perturbation in mean longitude.

R_a = range of variation of A.

R_{e^2} = range of variation of e^2.

Conclusion.

It has been pointed out that the process of reducing the disturbing function to one or more terms and then integrating is equivalent to picking out certain terms which arise at various stages in the ordinary method of successive approximations with regard to the disturbing masses and grouping them together.

Now the infinite series obtained by the ordinary methods are convergent for a certain range of values of the time $t_0 < t < T$ say, and the process will be legitimate if the sub-series taken out are also convergent. This would appear to be the case. We have, in fact, found approximate values for the sub-series.

But it must be observed that the formulæ obtained in this way are only valid for $t_0 < t < T$. For values of t greater than T we should be picking out a sub-series of terms from a divergent series; and even if the sub-series is convergent, it is hard to see how any meaning could be attached to the procedure.

T is probably very great. It may be infinite, but we do not know its value. We are not in a position to state, then, that the results obtained by the process described retain any significance for all time. Nevertheless, it is a process which has been used extensively in celestial mechanics, and the work of Laplace and Le Verrier on the secular perturbations of the major planets is based on it.

Bearing this reservation in mind, we may formulate the following general conclusions:—

1. The movements of asteroids whose mean motions are nearly half that of Mars are subject to long-period inequalities arising from the fact that their mean motions are nearly commensurable with that of Mars.

2. They are also subject to long-period inequalities ("secular" inequalities), due to the action of Jupiter.

3. The two principal long-period inequalities of the first kind arise from two terms in that part of the disturbing function due to the action

of Mars. One of these terms involves the eccentricity of the asteroid as a factor, and the other involves the eccentricity of Mars.

4. Long-period inequalities in the major axes arise from the cause mentioned in 1.

5. There are no long-period inequalities in the major axes arising from the action of Jupiter.

6. Long-period inequalities in the eccentricities arise from each of the causes mentioned in 1 and 2. The ranges of variation of those due to the action of Mars are appreciably less than those due to the action of Jupiter.

7. The mean longitude may be expressed in the form

$$L = n_1 t + \epsilon_1 + N_1 t + \text{periodic terms},$$

where n_1 is the initial value of the theoretical mean motion, this being deduced from the initial value of the major axis by Kepler's third law.

This may be written

$$L = Nt + \epsilon_1 + \text{periodic terms}.$$

N may conveniently be called the "real mean motion."

Of the periodic terms of long period in L the amplitudes of those arising from the action of Mars may be appreciably greater than the amplitudes of those arising from the action of Jupiter, and when the approach to exact commensurability of the mean motions is near enough, the phenomena of libration (the cases B in the theory) may manifest themselves, and the amplitude of the corresponding periodic term may be comparatively large. In fact, when k, the modulus of elliptic functions, tends to unity in any of the cases A and B, the amplitude A tends to $\frac{3}{4D_2} \times 90°$, *i.e.* to 90° approx. In the example above we found amplitudes of approximately 40° and 62°.

It is a pleasant duty to acknowledge my indebtedness to Professor Baker for his continued interest in this and the previous investigation.

Invisible Sunspots. By Dr. G. E. Hale.

(*Extract from a letter to Professor Newall,* 1921 *Dec.* 21.)

I have been able to give more time to my own solar work, and the measurement and reduction of the sunspot spectra, so long delayed by various causes, is now well advanced. Another piece of work which I had in mind when with you has also made progress recently.

You may remember that I commented on the strong tendency of spots to appear in the bipolar form, which is so marked that some 60 per cent. of all spots are double, while almost all single spots are followed (sometimes preceded) by streams of calcium flocculi. It struck me that an incipient spot, not dark enough to be visible, might frequently lie in the flocculi, at a point corresponding with the position of the second member of a full-fledged bipolar group.

Lately Ellerman and I have used the Zeeman effect to detect invisible vortices in such regions. Some time ago I found local magnetic fields in the midst of faculæ where no spots were visible, and it struck me that a systematic search for "invisible spots" might be worth while. The best way to find them is to mount a small half-wave plate above the compound quarter-wave plate and nicol ordinarily used over the slit of the spectrograph. The half-wave plate is made by a small electric motor to oscillate back and forth across the slit, thus transmitting alternately the red and violet *n* components of a Zeeman triplet. Promising regions of the sun, such as the calcium flocculi following a single spot, are moved slowly across the slit by the cœlostat motor, while the observer watches the iron line $\lambda 6173$ (the triplet used for our daily observations). The presence of a magnetic field with an intensity of 200 gausses, or even less, is shown by the widening of the line towards red or violet as the half-wave plate alternately covers and uncovers the slit.

The first local field thus found followed a single spot, and had the same polarity and approximate position as the following component of a regular bipolar group. (Near it was another local field of the same polarity.) This remained invisible for three days, and then appeared as an ordinary small-spot. On the return of this group in the next solar rotation this spot was no longer visible, but its field and another small field were detected as soon as the group had advanced far enough upon the disc. Two days later both of the small spots became visible. In another case the local field of a very small spot (the following member of a bipolar group) could be detected for two days after the spot had disappeared.

It will be interesting to go on with this work, and especially to look for local fields in isolated faculæ or flocculi where no spots can be seen. I am also anxious to see how long a magnetic field can be detected before or after the appearance or disappearance of a large spot. The effect irresistibly recalls the rise and fall of the free end of my flexible vortex near the water-surface as the speed of the motor varies, but in many cases, at any rate, a bipolar spot cannot be such a simple thing as a semicircular vortex ring. It certainly seems impossible to account in this way for the mixed polarities of the sunspot of last May (the one that produced the magnetic storm and aurora), and the proper motion of a vortex ring is always a serious objection to this hypothesis. Yet if the vortices of a bipolar group penetrate to a sufficient depth they must certainly unite to form a half-ring.

Another thing that interests me is the bearing of invisible spots on the level of origin of the main spot vortex. The calcium (H) flocculi now appear to be the best guide to the existence of invisible spots, but it is probable that faculæ would be equally useful if as easily observed near the centre of the sun. "Invisible spots" may sound like a misnomer, but it seems the simplest name to apply to spots which may be visible during part of their existence.

Terrestrial Magnetic Disturbances and Sun-spots.

By Rev. A. L. Cortie, S.J., F.Inst.P.

One of the chief conclusions stated by Mr. Maunder in his very important paper on "Magnetic Disturbances, 1882 to 1903, as recorded at the Royal Observatory, Greenwich, and their Association with Sun-spots" (*Monthly Notices R.A.S.*, 65, 23), was to the effect that "the rotation periods given us by the magnetic disturbances not only agree in the mean with the rotation periods given by sun-spots, but the limits within which they vary are the same." The object of the present paper is to extend the investigation to some longer magnetic series of the present solar cycle, and to bring evidence to show that the connection is still closer. In fact, the duration of the magnetic period is in general a function of the latitude of the spot-disturbance. This paper has grown out of a correspondence on the subject in *Nature* (1921 Oct. 6, Oct. 27, Dec. 29; 1922 Jan. 12). It will not be necessary to set out in detail the lengthy series of magnetic disturbances upon which it is founded. These can be reconstructed, from the limiting dates in the third column of the subjoined table, by means of the lists of magnetic disturbances given in the Annual Reports of this observatory.

TABLE.

Terrestrial Magnetic Disturbances and Sun-spots.

	Year.	Duration.	No. of Days.	Mean Period. Synodic.	Mean Period. Sidereal.	Solar Daily Rotation.	Solar Latitude.	Spot Group.
1	1917	July 12–Nov. 26	137	27·40	25·49	14°·13	±15·8	+16·0
2	1918	July 1–Jan. 8	218	27·25	25·36	14·20	±13·5	+16·1
3	1919	May 24–Oct. 6	135	27·00	25·14	14·32	± 5·7	− 7·8
4	1920	Jan. 1–Nov. 21	325	27·08	25·21	14·28	±10·2	− 5·8
5	1921	March 22–Sept. 29	191	27·29	25·39	14·18	±14·1	+ 9·0
6	1921	May 21–Dec. 24	217	27·13	25·25	14·26	±10·9	?
		Mean sidereal period,			25·31			

For the sun's mean period of rotation Carrington's value is 25·38, and Spoerer's 25·23. The mean of these two numbers is 25·31, coinciding exactly with the mean sidereal period of these magnetic disturbances.

Notes.

1. The associated sun-spot group was the great group of August, recurring in September, Nos. 771, 813 of the Tortosa Boletín Mensuel.

2. The associated sun-spot group, which recurred three times, bears the numbers 1045, 1066, 1088, 1106 of the Tortosa Observatory records.

3. The associated sun-spot groups are 1242, 1262, 1282, 1300, four appearances.

4. The associated sun-spot group appeared six times, Nos. 1370,

1376, 1392, 1405, 1416, 1425. Mr. Evershed has discussed this series (*Nature*, 1921 Dec. 29).

5–6. These two series are most probably due to the equatorial group 1921 May 8–19 (*Monthly Notices*, 81, 516). The magnetic storms of May 12–18 and 19–21 were so protracted that the recurrences cover several successive days, and so lengthen the period. With regard to No. 5, the fairly extensive spot of March 22 was in latitude $+9^{\circ}$.

The following conclusions can be drawn from this discussion:—

First: the parity of the solar rotation periods and those of the successive magnetic sequences would demonstrate, as Mr. Maunder contended, that the magnetic phenomena are associated primarily with the sun-spots. Other solar appearances as flocculi, faculæ, and H_{α} absorption markings may be concomitant with sun-spots, but they are subsidiary to them.

Second: it seems possible to extend and amplify Mr. Maunder's more general conclusion as to the periodicity of the magnetic disturbances agreeing, in the mean and in their limits, with the rotation period of the sun, and to state that, in general, the duration of the magnetic period in a series of disturbances is a function of the solar latitude of the associated sun-spot period.

Stonyhurst College Observatory:
1922 *January* 9.

Centres of Sun-spot Activity during the Year 1917.

By H. W. Newton. (Plate 3.)

(*Communicated by the Astronomer Royal.*)

In the *General Catalogue* of the Greenwich Heliographic Results for 1917, now being passed through the press, an attempt has been made in the "Remarks" column to co-ordinate the various centres of sun-spot activity. The year 1917, being the maximum year of the present cycle, contains much material for discussion. During the year there were recorded 52 series of groups of spots termed "Recurrent," *i.e.* groups which were seen during at least two successive rotations of the Sun, with apparently continuous activity. Besides these "Recurrent" groups, there were a large number which have been termed "Revivals" in previous volumes of the Greenwich Results. In these cases it was certain that, although a spot group occupied the position of that of another seen either in the preceding or succeeding rotations, yet there had been a definite break in the continuity—either the formation or disappearance of one or of both groups had been observed to take place on the Sun's disc. Groups related in this way have been collected in a table given in the *Introduction* for 1917. They form 53 series containing in all some 115 groups, and these are seen to have certain characteristics to which attention is drawn in this note.

1. The close agreement in position in succeeding rotations between the centres of groups which form a series is a noticeable feature. To

illustrate this, a short extract is made from the table in which they are collected in the *Introduction*.

Group No.	Mean Position. Long I.	Long. II.	Lat.	First Seen. Date.	Long. from C.M.	Last Seen. Date.	Long. from C.M.	Mean Area.
	°	°	°		°		°	
7968	99	100	+22	Jan. 28	−67	Jan. 31	−33	9
7999	101	102	+21	Feb. 24	−71	Mar. 5	+44	100
8020	13	3	+13	Mar. 12	+46	Mar. 14	+78	23
8045	11	359	+15	Apr. 6	+21	Apr. 10	+71	17
8071	8	359	+17	May 3	+11	May 8	+76	60
8093	13	359	+16	May 23	−76	June 4	+79	138
8170	253	226	+14	July 25	−83	Aug. 5	+59	39
8216	256	232	+16	Aug. 21	−84	Sept. 1	+63	34
8243	37	347	+10	Sept. 7	−81	Sept. 18	+69	104
8286	45	348	+10	Oct. 12	+27	Oct. 15	+72	38
8307	55	347	+ 8	Nov. 1	−59	Nov. 11	+79	229
8334	66	347	+ 6	Nov. 26	−74	Dec. 6	+57	77

Longitude I. is computed according to Carrington's assumed rotation period of 25·38 days.

Longitude II. according to a rotation period varying with the latitude (E. W. and A. S. D. Maunder).

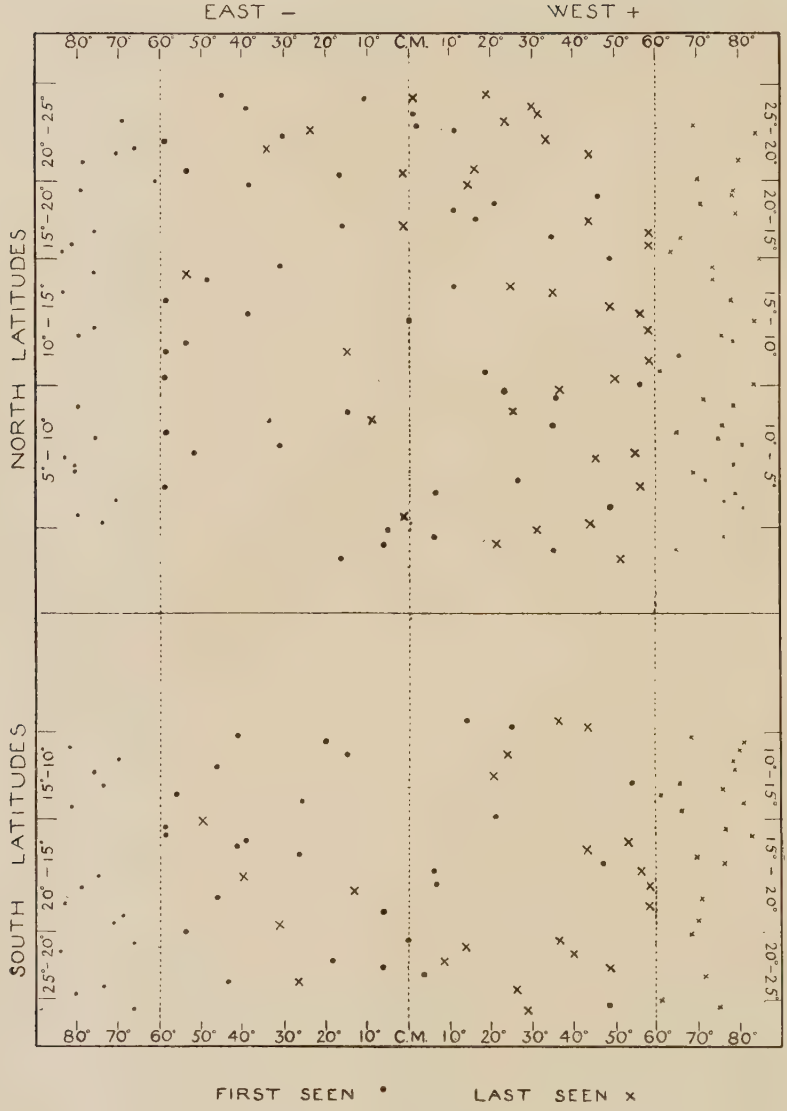
Longitudes from Central Meridian: − indicates east, + indicates west.

Areas are given, corrected for foreshortening, in millionths of the Sun's visible hemisphere.

Thus Group 7968 was first seen near the east limb on Jan. 28, but disappeared by Feb. 1 when about 30° east of the Central Meridian. The group had re-formed when the region was next visible at the east limb and continued till about 45° west of the Central Meridian. Again, Group 8020 appeared on March 12 west of the Central Meridian and disappeared round the Sun's limb after March 14. It was not seen at the return of that region to the east limb, but reappeared when some 20° west of the Meridian on April 6. This likewise passed round the west limb but did not reappear at the east limb. On May 3, however, the group had re-formed, after which the activity was apparently continuous till the region was last seen at the west limb on June 4.

These may be taken as typical examples. They show how frequently a disturbed region may exist in the Sun whose presence is only intermittently indicated by spots.

2. The frequency of distribution in latitude of the "Revival" groups for 1917 shows a marked want of symmetry with regard to the solar equator. There were twice as many of these groups in the northern as compared with the southern hemisphere. It should be noticed that the excess in the northern hemisphere is almost entirely due to several groups situated between latitudes 4° and 8° N.; no "Revival" groups appeared in the southern hemisphere nearer the equator than 8°. The



Longitudes measured from the Sun's Central Meridian at which Spots are seen to form and disappear ("Revival" groups of 1917).

general sun-spot activity for 1917 is divided north and south of the equator in the proportion of 56 to 44 respectively. As the centres of more prolonged activity are being considered in this paper, this marked excess of the northern hemisphere over the southern confirms several indications, which have been pointed out, of some salient differences between the two hemispheres. It may be remarked that for the last four cycles the maxima of the northern spotted areas have been increasingly higher.

3. The most interesting characteristic of these "Revival" groups is illustrated by the accompanying diagram in which are plotted the positions of each group, relative to the Central Meridian, when they were first and last seen respectively. The discussion is limited solely to the actual formations and extinctions of these groups of spots, and therefore only those points in the diagram are to be considered which lie between the Central Meridian and 60° east and west of it. The majority of the points at longitudes greater than 70° from the Central Meridian arise, of course, from groups passing round the Sun's limb. It is, however, quite certain that, as the daily photographic record is complete, all the points at longitudes not greater than 60° from the C.M. represent *actual* formations and extinctions of spots. This is confirmed by the following table, derived from a number of spot groups, from which it is evident that in practice a small group just forming or disappearing can be detected on the photographs as readily in the outer zones, not exceeding 60°, as in those nearer the C.M.

Average Area (corrected for foreshortening) of Spot Groups when first and last seen.

Long. from C.M.	0°—10°.	10°–20°.	20°–30°.	30°–40°.	40°–50°.	50°—60°.
First seen	23	17	19	19	20	22
Last seen	7	7	10	8	9	11

Referring to the diagram, it is apparent that (*a*) formations of "Revival" groups tend to take place east of the Central Meridian, (*b*) extinctions most frequently occur west of the Central Meridian. Actually, of the 73 formations, 43 occurred east of the C.M.; of the 57 extinctions, only 13 took place east of the C.M. In zones of 10° solar longitude they are grouped as follows:—

Longitude from C.M.		Formations.	Extinctions.
East	60°–50°	13	1
	50–40	7	1
	40–30	8	3
	30–20	3	2
	20–10	7	2
	10–0	5	4
West	0–10	8	2
	10–20	6	4
	20–30	5	8
	30–40	4	9
	40–50	5	9
	50–60	2	12

These results are strongly suggestive that the formations and extinctions of "Revival" groups are dependent on the manner in which the Sun is presented to the Earth, for the Central Meridian has no significance except in relation to the Earth. This feature, however, Mrs. Maunder * has already shown to be characteristic of all spots considered as a whole. From a thorough discussion of material, including the numbers and areas of all spot groups in the cycle 1889–1901, she discovers "a well-marked and steady preponderance of the eastern half of the Sun's disc over the western half, both as to the areas of the spots and as to the numbers of the separate groups." This phenomenon, shown also by the faculæ † and by prominences in general, ‡ suggests, as has been pointed out, a damping effect on spot formation exerted by the Earth, though it is difficult to see how this can arise. It is of interest that this apparent effect is shown in a marked way by the "Revival" groups. Firstly, they are indicative of some of the main centres of activity. Secondly, they may be considered as representing latent spot activity for part of their life-history. If any Earth effect exist, it is probable that it would be operative on them in a greater degree than on groups of more intense and continuous activity. It is desirable to extend the examination of "Revival" groups to other years, and this note is therefore offered tentatively until a fuller discussion can be undertaken.

Royal Observatory, Greenwich:
1922 *January* 4.

Solar Motion from 1922 *Radial Velocities.*

By George Forbes, M.A., F.R.S.

Voûte's First Catalogue of Radial Velocities bears evidence of sound judgment in its compilation, and, in any case, it affords an opportunity for deriving the amount and direction of the solar motion based upon a greater number of stars than any previous attempt.

A preliminary survey of + and − values greater than the solar velocity is very suggestive. Omitting the 138 nebulæ, ten clusters, and one star (Barnard's) with "very discordant values," there remain 1922 radial velocities, out of which 697 have velocities greater than a solar motion of 20 km./sec. Any concentration of +'s and −'s among these stars must therefore be quite independent of a solar motion of 20 km./sec. or less. A chart of these 697 high-velocity stars shows a great concentration of + radial velocities from galactic longitudes 110°, through 180°, to 270°. It contains a large oval patch about 90° diameter, containing 197 plus, and only four minus, stars. The opposite hemisphere is full of minus stars with very few plus ones. It also contains an oval patch, even more condensed, containing 205 minus, and only 19 plus, stars. The centre of minus concentration may be at galactic longitude 30°, latitude + 10°. The centre of plus concentration in galactic latitude − 10° to − 20°, longitude uncertain, perhaps

* *M.N.*, 67, 460 and 471. † *M.N.*, 80, 738.
‡ *M.N.*, 67, 472. *Kodaikánal Bulletin*, Nos. 28, 35, 67.

180°. These concentrations are not affected by a solar motion of 20 km./sec., yet they indicate a general stream of stars, Kapteyn's stream I. in fact. The conclusion is that the prevalent motion is better described as a stream of stars, with every grade of velocity, than as a controlling solar motion, and that the solar velocity derived from any observations will be found to vary with every list of stars used for the purpose.

In further illustration of the tendencies exhibited by high-velocity stars, a free curve can be drawn on the chart, dividing the sky into two nearly equal parts. There is only one part of the boundary line where pluses and minuses are mixed, and 34 pluses appear in the minus hemisphere. Neglecting these local, boundary exceptions, 607 of the stars, having radial velocities over 20 km./sec., are travelling in directions from the minus hemisphere to the plus hemisphere; only 56 in the opposite direction. Now, this excess is absolutely independent of any solar velocity of 20 km./sec. or less. Table I. gives all the information required for further examination of this point.

In deducing the solar motion from this catalogue of radial velocities, the method used by Campbell in his *Stellar Systems* has been followed exactly with one exception:—no rejection of any stars has been adopted with the hope of thus attaining a closer value of the solar motion *relatively to the centre of gravity of our sidereal system.* Any such rejection must be arbitrary, and is not necessarily conducive to the purpose with which the rejection is made.

Galactic co-ordinates are used throughout with galactic pole at R.A. $12^h\ 40^m$; n.p.d. 62°; 0° long. at $18^h\ 40^m$. Table II. shows the algebraic mean radial velocity and number of stars for each section ($20° \times 20°$ except at the poles) in one hemisphere, together with those for the corresponding section in the other hemisphere (signs reversed).

w = weight attached to each section. = number of stars in the section.
α = galactic longitude of the section.
δ = galactic latitude of the section.
V = mean radial velocity in that section.

$A = \sqrt{w} \cdot \cos\alpha \cdot \cos\delta$. $\quad B = \sqrt{w} \cdot \sin\alpha \cdot \cos\delta$
$C = \sqrt{w} \cdot \sin\delta$. $\quad D = \sqrt{w} \cdot V$.

72 equations resulted of the form

$$Ax + By + Cz = D,$$

where

$$x = V_0 \cdot \cos\alpha_0 \cdot \cos\delta_0,$$
$$y = V_0 \cdot \sin\alpha_0 \cdot \cos\delta_0,$$
$$z = V_0 \sin\delta_0,$$

V_0, α_0, δ_0 giving the solar motion relative to 1922 stars included in the investigation.

From these 72 equations the usual three normal equations of the method of least squares were deduced. Their solution gave

$V_0 = 22{\cdot}00$ km./sec.
$\alpha_0 = 20°\ 50'$;
$\delta_0 = +20°\ 15'$;
} or R.A. $= 18^h{\cdot}0 = 270°$. Decl. $= +27°$.

TABLE I.—High Velocity Stars. (Over 20 km./sec.)

Average + positive radial velocities, and number of stars.

Galactic Longitudes.

Galactic Latitude.	280°.		260°.		240°.		220°.		200°.		180°.		160°.		140°.		120°.		100°.		80°.		60°.		40°.		20°.		360°.		340°.		320°.		300°.		280°.	
+90°																																				
+60	51	(1)	—		—	47	(2)	—	—	—	—	—	—	—	—	65	(1)	24	(1)	—	—	—	—	—	—	26	(2)	—	—	—	—	—	—	—	22	(1)
+40	—				—	—	—	69	(2)	30	(3)	46	(5)	29	(3)	25	(2)	23	(1)	—	—	—	—	—	—	—	—	—	30·5	(2)	52·7	(3)	—	—	—		
+20	—	—	28		(2)	38	(5)	26	(5)	34	(11)	23	(4)	30	(8)	40	(5)	—	—	—	—	29	(4)	—	—	—	—	—	34	(2)	98	(1)	—	—	31·7	(3)	
0	25	(3)	24		(1)	31	(5)	31	(8)	56	(4)	34	(6)	28	(8)	44	(3)	—	—	20·5	(2)	29·7	(4)	—	—	—	—	27	(1)	68	(2)	37	(1)	—	—	24	(2)
-20	26·6	(5)	26·4	(13)	41·3	(10)	39	(14)	39·7	(10)	31	(21)	29·7	(7)	38·6	(5)	35	(3)	—	—	—	—	—	—	—	34	(1)	—	—	25	(1)	22·3	(3)	26	(2)	—	—
-40	—	—	32		(9)	26	(8)	83	(9)	61	(7)	26	(5)	58	(15)	35	(2)	58	(2)	29	(1)	—	—	—	—	33	(1)	—	—	22	(3)	28	(3)	67	(4)	60	(2)
-60	24	(1)	40		(9)	28	(3)	39	(5)	28	(4)	45	(3)	50	(5)	47	(5)	—	—	—	—	45	(1)	—	—	40	(2)	—	—	33	(1)	31	(2)	26	(1)	26	(1)
-90	—	—	37		(2)	27	(1)	20	(2)	20	(1)	25	(2)	—	—	—	—	29	(1)	—	—	—	—	—	—	—	—	22	(2)	—	—	25	(2)	26	(2)	—	—

Average − negative radial velocities, and number of stars.

+90°																																				
+60	35	(1)	—		—	29	(2)	—	—	29	(1)	—	—	39	(4)	97	(1)	—	—	—	—	—	—	24	(1)	39·3	(3)	41·5	(2)	—	—	39	(1)	24·5	(2)	54	(1)
+40	37	(2)	34		(1)	57	(1)	25	(1)	29	(2)	—	—	24	(1)	28	(2)	—	—	52	(3)	46	(1)	30·3	(3)	69·6	(8)	29·9	(7)	39·4	(5)	70	(1)	70	(1)	29·5	(2)	
+20	—	—	22		(2)	—	—	—	—	—	—	—	—	138	(2)	35	(2)	28	(2)	30	(2)	28	(5)	31	(4)	49·7	(13)	29·0	(8)	61·7	(8)	55·5	(6)	24	(3)	39	(1)	
0	45	(2)	—		—	—	—	—	—	—	—	—	—	25	(2)	44	(3)	33	(3)	65	(5)	32·6	(12)	43·5	(13)	34·9	(28)	40·1	(11)	46·6	(5)	38·6	(5)	28·9	(9)	42·7	(4)	
-20	22	(1)	—		—	—	—	—	—	39	(1)	—	—	190	(1)	23	(1)	35·7	(4)	62·4	(14)	29·1	(8)	46	(10)	30·8	(10)	43·4	(8)	38	(5)	33·7	(4)	28	(4)	30	(1)	
-40	—				—	—	—	—	—	34	(1)	42	(1)	—	—	—	—	49	(4)	46	(6)	30	(3)	37	(3)	44	(5)	21	(1)	72	(4)	83	(2)	30	(1)	22	(2)	
-60	—	—	—		—	—	—	—	—	21	(1)	—	—	23	(2)	41	(1)	34	(1)	33	(3)	30	(2)	39	(2)	—	—	55	(2)	32	(2)	30	(1)	38	(1)	—	—	
-90	—	—	—		—	30	(1)	—	—	—	—	20	(1)	—	—	33	(1)	41	(2)	29	(1)	—	—	31	(2)	26	(1)	—	—	—	—	—	—	—	—	—	—	

TABLE II.—All stars, excluding clusters and nebulæ. Average radial velocities (+ and −) for each 20° × 20°, including radial velocities (− and +) in the corresponding section in opposite hemisphere; and number of stars in each section.

Mean Galactic Longitude.

Mean Galactic Latitude.	270°.	250°.	230°.	210°.	190°.	170°.	150°.	130°.	110°.
+70°	− 3·5 (19)	− 6·6 (9)	+ 8·5 (15)	+ 1·9 (14)	−14·4 (10)	− 4·0 (7)	−15·7 (16)	−11·1 (13)	+0·9 (11)
50	+ 0·6 (23)	− 1·0 (15)	+ 0·9 (17)	+ 2·8 (17)	+10·2 (18)	+12·1 (21)	+ 2·2 (31)	− 0·2 (23)	+0·4 (14)
30	+ 9·7 (25)	+ 5·7 (23)	+13·4 (23)	+14·0 (26)	+18·1 (24)	+13·3 (29)	+ 2·6 (30)	− 0·2 (24)	+4·7 (27)
10	+14·9 (60)	+11·2 (40)	+19·5 (35)	+14·4 (48)	+22·4 (27)	+ 9·4 (38)	+ 7·5 (44)	+ 1·1 (40)	− 3·7 (30)
−10	+14·9 (38)	+14·1 (55)	+22·5 (53)	+25·4 (68)	+21·6 (41)	+19·0 (49)	+ 6·3 (48)	+12·5 (46)	+2·2 (45)
30	+ 5·4 (17)	+11·5 (31)	+15·3 (33)	+47·4 (31)	+22·3 (26)	+19·1 (32)	+28·0 (44)	+10·1 (30)	− 5·5 (38)
50	+14·1 (19)	+18·1 (25)	+13·8 (16)	+35·4 (26)	+16·4 (21)	+17·2 (20)	+ 6·5 (24)	+19·7 (18)	+4·0 (19)
70	+ 3·3 (10)	+12·2 (13)	+ 6·0 (9)	+20·6 (11)	+12·1 (13)	+ 3·3 (12)	+ 7·8 (9)	+18·4 (7)	+3·6 (12)

The Slide-Rule as an Auxiliary to the Solution of Kepler's Equation.
By Major A. E. Levin.

In the following method the ordinary slide-rule is used to give, by direct reading, a first approximation, which is usually within two or three minutes of the rigorous value. It may also be used with advantage in the subsequent corrections.

(*a*) *Approximate Solution.*—The equation, with M and E both expressed in degrees, takes the form $M = E - 57{\cdot}29 \ldots e \sin E$. If then the slider be reversed, so as to have the trigonometric scales uppermost and the right-hand end of the sine scale be set on the value $57{\cdot}29 \ldots e$, previously found either by logarithms or by ordinary multiplication on the front of the slide-rule, we shall have, for any assumed value for E on the sine scale, the corresponding value of $57{\cdot}29 \ldots e \sin E$ on the upper scale. It is then easy to pick out a value of E such that the readings of the two scales differ by the given value of M. In practice it is convenient to effect the setting directly by the trigonometric scale by employing the approximation $57{\cdot}29 \ldots e = \frac{e}{\sin 1^\circ}$ (correct to 1 in 19700). Or, for values of e above 0·5, we may set 2° on the sine scale against $2e$ (formed mentally), which gives an approximation still within the limits to which the scale can be read. This latter method has the advantage that the right-hand end of the slider is within the limit of the upper scale, so that readings up to E = 90° and above may be taken at a single setting.

If the eccentricity be given by ϕ, the eccentric angle, the scale may be set by putting $\sin 1^\circ$ against 1 of the fixed scale and reading $\frac{\sin \phi}{\sin 1^\circ} = 57{\cdot}29 \ldots e$ by means of the cursor, afterwards setting the end of the sine scale to the position so held.

For values of E near to 90° the sine scale is so cramped that it would appear hopeless to read to 1°, much less to 0°·1. But since the upper scale on which 57·29 . . . . $e \sin E$ is read remains open, it is possible, as soon as a very rough estimate of E is formed, to read off 57·29 . . . . $e \sin E$ to 0°, and by adding it to M to obtain E.

For values of E above 90° the equation may be transformed to $M' = E' + 57{\cdot}29 \ldots . e \sin E'$, where M′ and E′ are the supplements of M and E respectively.

The method is considerably quicker than the use of the various formulæ for an approximate solution, and, compared with graphical methods such as that of Waterston, has the advantage that a whole series of readings can be taken from one, or at most two, settings of the rule, while for high values of e and low values of M it gives much closer results owing to the openness of the lower parts of the logarithmic scale.

The residual errors being dependent not so much on theoretical considerations as on the practical accuracy of the division of the slide-rule and the power of accurately subdividing the scales by the eye, a practical test has been made as follows:—An ordinary 25-cm. slide-rule was used. In the lower values the division of degrees to minutes instead of decimals is somewhat inconvenient, and it is much to be desired that some maker will adopt the decimal subdivision of degrees in the trigonometric scales. Ten values of e between 0 and 0·85 were taken as representing low and medium eccentricities, and for each one ten different values of E were read off from arbitrary values of M. Ten values of e between 0·85 and 1·00 were taken and similar readings taken. Both for e and M values with many decimals were used, so as to give full effect to the error due to discarding all after the third or fourth in the slide-rule setting. From each value of E so found a corresponding M was computed by seven-figure logarithms, the residuals ΔM found, and from these, by the factor $\frac{1}{1 - e \cos E}$, the values of ΔE. The p.e. was then computed for each value of e and also for each group—medium and high eccentricities. The results are shown in the following table:—

e.	p.e. of M.	p.e. of E.	e.	p.e. of M.	p.e. of E.
	′	′		′	′
0·0772727	0·17	0·17	0·8636364	2·10	2·06
·1545455	0·35	0·36	·8772727	1·10	3·52
·2318181	0·95	0·88	·8909091	2·10	3·69
·3090909	0·64	0·73	·9045454	2·10	3·90
·3863636	2·14	2·14	·9181818	1·43	3·74
·4636364	0·88	1·00	·9318182	3·30	7·50
·5409091	1·66	2·07	·9454545	2·18	4·40
·6181818	1·50	1·65	·9590909	1·63	2·68
·6954545	0·89	0·81	·9727273	2·28	5·86
·7727273	1·05	1·10	·9863636	2·28	3·65
Combined	1·17	1·26	Combined	2·13	4·34

It will be seen that the p.e., though naturally increasing with high eccentricity, is still within reasonable limits even at the highest values of e chosen. Further, the fact that the open scale gives the closest readings for low values of E where the correction factor $\frac{1}{1-e\cos E}$ is large makes the p.e. of E not much greater than that of M – in fact, in one or two cases it is slightly less.

(*b*) The correction of the approximate value of E found as above by means of the correction factor $\frac{1}{1-e\cos E}$ may also be very conveniently effected by the slide-rule. Setting the right-hand end of the sine scale on e, the values of $e\cos E$ are given for various values of E, or by reading the scale backwards the complement to 1 may be read direct. For values of E near to 90° the reading will be in the left half of the upper scale, so that the complement will begin with a 9, following which three figures can be easily read. For values of E above 90° the numerical value of $e\cos E$ is added to 1. For low values of E, when the sine-scale is cramped near 90° and an accurate reading is specially important owing to the high value of $\frac{1}{1-e\cos E}$, four- or five-figure accuracy may be attained by employing the identity $1-e\cos E = 1-e+2e\sin^2\frac{E}{2}$. Here $1-e$ is a constant for all values of M, and is known to as many figures as are given for e. The remaining term is read by two settings of the rule, using the trigonometrical scales only, multiplying e first by $\frac{\sin\frac{E}{2}}{\sin 30^\circ}$, and then again by $\sin\frac{E}{2}$. With the lower values of E the result, being preceded by one, two, or three noughts, will give a value of $1-e\cos E$ to four or five figures.

In dividing ΔM by $1-e\cos E$ increased accuracy may be attained in those cases where $1-e\cos E$ is near to 1 or to 0·1 by the identity $\frac{\Delta M}{1\pm n} = \Delta M \mp \frac{n\Delta M}{1\pm n}$. That is to say, after setting in the ordinary way to divide ΔM by $1-e\cos E$ and reading off the result as nearly as possible, the last figure may be checked, or another added, by reading at the same setting of the rule, the small correction $\frac{e\cos E\Delta M}{1-e\cos E}$ to be mentally subtracted from or added to the value of ΔM. In this way a correction at least equal in accuracy to the table given in Bauschinger's *Tafeln zur theoretischen Astronomie* is attained, without the necessity of interpolation. Further accuracy in the value of $1-e\cos E$ is useless owing to the existence of second-order terms. In fact, in cases where ΔE is relatively large, further steps in approximation may be saved by taking in another term of the Taylor series:—

$$\Delta E = \frac{\Delta M}{1-e\cos E} - \frac{e\sin E}{(1-e\cos E)^3}\frac{(\Delta M)^2}{2}.$$

That is, after computing $\frac{\Delta M}{1 - e \cos E}$, we obtain a further small correcting term by multiplying it, at three settings of the rule, by

$$\frac{e}{(1 - e \cos E)^2} \times \frac{\Delta M}{2 \times 57^{\circ}3} \times \sin E.$$

The use of this correcting term when e or ΔE is large and E small will usually lead to the desired accuracy being reached at a single correction.

In *A.N.*, 186, No. 4462, Mache describes a slide-rule solution of Kepler's equation. In it he does not turn the slider with the trigonometrical scales uppermost, so that it is necessary to compare a reading on the front with one taken through the "window" at the back instead of readings of juxtaposed scales as in the above method. Moreover, the slider has to be set for each value of M instead of a series of readings being taken from one setting.

Parallax of B.D. 80°694. By C. R. Davidson.

(*Communicated by the Astronomer Royal.*)

In the course of the determination of the parallax of Lalande (F) 3880 (81°742) it was noticed that one of the comparison stars (80°694) had an appreciable proper motion.

This star was rejected as a comparison star and its parallax determined.

The details are:

B.D. 80°694 R.A. $21^h\ 24^m\ 24^s$ Dec. +81° 10′ (1900·0) Vis. mag. $9^m{\cdot}5$, Photog. $10^m{\cdot}1$

μ_α	$-''{\cdot}056$	μ_δ	$-''{\cdot}036$ from meridian observations *
$\mu_{rel.}$	$-''{\cdot}078$	$\mu_{rel.}$	$-''{\cdot}049$ from the photographs
π	$+''{\cdot}045 \pm ''{\cdot}007$		

from which an absolute magnitude $7^m{\cdot}9$ is deduced.

The parallax is remarkable on account of the smallness of the proper motion, whose direction is away from the antapex of the solar motion. A velocity of about 27 km. per second across the line of sight is indicated.

* *Greenwich Astrographic Catalogue*, 4, 34.

Note on the Masses of the Stars. By Henry Norris Russell and David L. Webster.

One of the most beautiful features of Eddington's theory of the internal constitution of the stars is the proof that the masses of the stars are determined (as regards order of magnitude) by fundamental constants of nature. A gaseous mass can shine as a star only when the pressure of radiation in its interior is a sensible fraction of the whole pressure: but if this fraction is too great, the mass tends to break up into smaller stars.

The equation from which this relation may be derived has been put in an elegant form by Jeans.* For a gas which obeys Boyle's law, this may be written

$$\frac{(1-\beta)}{\beta^4} = \frac{\pi}{18}\alpha G^3\left(\frac{m}{R}\right)^4 M^2\theta^3\left(\frac{\bar{\rho}}{\rho}\right)^4 \quad . \quad . \quad . \quad (1)$$

where $1-\beta$ is the ratio of the radiation pressure to the whole pressure at any point within the star, α the energy density of radiation in an enclosure at 1° Abs., G the constant of gravitation, m the mean molecular weight of the gas (taking account of ionisation), R the gas constant, M the star's mass, $\bar{\rho}$ the mean density, ρ the density at any point, and θ the ratio of the whole pressure at this point to the gravitational pressure at the centre of a homogeneous sphere of the same size and mass. If N is the number of molecules in a gram-molecule, k the gas constant for one molecule, and μ the average mass of one molecule, we have $R = Nk$, $m = N\mu$; moreover, in accordance with Planck's theory, $\alpha = \frac{8}{15}\frac{\pi^5 k^4}{c^3 h^3}$, c being as usual the velocity of light and h the quantum constant.

Introducing these into (1), we find

$$\frac{1-\beta}{\beta^4} = \frac{4\pi^6}{135}\frac{G^3\mu^4 M^2}{c^3 h^3} \quad . \quad . \quad . \quad . \quad (2)$$

This may be further simplified by introduction of the theory of "ultimate rational units" proposed by Lewis and Adams.† According to this,

$$c^3 h^3 = \frac{2^{15}}{15}\pi^{11} e^6 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where e is the electronic charge (in electrostatic units). This permits us to write

$$\frac{1-\beta}{\beta^4} = \frac{1}{9\cdot 2^{13}\cdot \pi^5}\left(\frac{M}{\mu}\right)^2\left(\frac{G\mu}{e^2}\right)^3\left(\frac{\theta^3\bar{\rho}^4}{\rho^4}\right) \quad . \quad . \quad . \quad (4)$$

The quantities in brackets which appear in this equation admit of simple physical interpretation. The first is *the number of molecules in*

* *M.N.*, 69, 322 (1919).

† *Physical Review*, series 2, 3, 92 (1914). Whether this theory be accepted or not, the fact remains that $\frac{ch}{e^2}$ is a pure number, and has very nearly the value given by (3).

the star, and the second *the ratio between the gravitational attraction between two molecules and the electrostatic repulsion between two electrons at the same distance.* The third is a pure number, depending on the internal constitution of the star, and will in general vary from point to point.

Upon the assumptions of Eddington's theory, this quantity will be a constant all through the star, and equal to 0·091831 (as is easily found from Emden's data,* which show that at the centre of such a mass (his "polytrope" with $n=3$) $\frac{\rho}{\bar{\rho}}=54\cdot360$, and $\theta=92\cdot90$). These numbers were derived by solving a differential equation by means of quadratures, and it is not known whether or not they are expressible in terms of well-known constants such as π.

Introducing this in (4), we find

$$\frac{1-\beta}{\beta^4}=4\cdot070\times10^{-9}\left(\frac{M}{\mu}\right)^2\left(\frac{G\mu^2}{e^2}\right)^3 \quad . \quad . \quad . \quad (5)$$

which applies to all giant stars on Eddington's theory. Now let R be the radius of the star, and r that of a sphere of volume equal to that occupied, on the average, by a single molecule. Also let μ' be the mass of such a molecule, measured in the same way that electric charge is measured (*i.e.* so that the gravitational force between unit masses at unit distance has unit value (one dyne)). Then $\mu'^2=G\mu^2$, and we may write (5) in the form

$$\frac{R}{r}=25\cdot03\frac{e}{\mu'}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad (6)$$

The factor involving β has the value 0·18 for a mass equal to the sun. For masses $\frac{1}{8}$ and 20 times the sun's (including the whole range of known stellar masses), the corresponding values are 0·0028 and 72, and we have from (6) for these limiting values $\frac{R}{r}=9\cdot4$ and $51\frac{e}{\mu'}$. The average distance between molecules in the interior of the star will be slightly less than $2r$. Hence we may say:

A mass of gas will appear as a star provided that the ratio of the radius of the mass to the average distance between the molecules which compose it is between five and thirty times the ratio of the unit electric charge to the average mass of a molecule (this mass being measured in the same manner as the charge).

If the internal constitution of the star follows some other model than Eddington's, the numerical values of this ratio will differ somewhat, but will be of the same order of magnitude. (Even in the extreme case of a star of uniform density they would be only 50 per cent. greater.) Moreover, although the observed masses of the stars have been utilised, at the end, to fix these limits, a mere consideration of the magnitude of the radiation pressure would have led us to results of the same order of magnitude, irrespective of any acquaintance with actual stars.

The conclusion stated above appears therefore to be a general one.

Princeton University Observatory:
1921 *October* 11.

* *Gaskugeln.*

The Number of Stars of Different Magnitudes in the Hyderabad Astrographic Catalogue. Fourth Paper. Zone −20°. By T. P. Bhaskaran, M.A.

1. References to former papers on this subject are *M.N.*, **77**, 432; **78**, 367; and **80**, 623. The fourth volume of the Hyderabad Astrographic Catalogue, containing the measured co-ordinates and diameters of 79,590 star-images in Zone −20°, is now published. As in the case of previous zones, the numbers of stars recorded under different diameters have been counted, and the results are given in the present note.

2. The work on this Zone was divided into three or four sections, taken and measured at different times, with considerable intervals between, the whole extending over nearly three years. The exposures had to be varied with the quality of the plates and also with the time they had been kept in stock before use; but within each group they have remained fairly constant.

3. Table I. gives the cumulative totals of the number of stars counted in each hour of R.A. Under the heading N_{50} are given the numbers of stars whose recorded diameters are greater than 50 (in units of 0″·15). The number of stars with diameters greater than 30

TABLE I.

Cumulative Totals for Ten Plates in Each Hour of R.A. Zone −20°.

R.A.	N_{50}.	N_{30}.	N_{20}.	N_{12}.	N_8 or All.	Gal. Lat.
h						°
0	82	444	899	1417	1628	−82·0
1	90	526	991	1557	1717	−76·5
2	69	385	725	1279	1601	−64·5
3	47	500	971	1593	1947	−51·0
4	62	562	1196	2226	2649	−37·5
5	116	751	1914	3623	4406	−24·5
6	135	1424	3431	6168	7142	−11·5
7	183	2561	6603	10703	11627	+ 1·0
8	171	1174	2640	5941	6465	+12·5
9	249	1407	3144	6119	6924	+23·5
10	210	1356	2605	4435	5042	+33·0
11	147	1019	2037	3083	3447	+39·5
12	119	711	1409	2374	2746	+42·0
13	34	466	1130	2054	2333	+40·5
14	56	548	1224	2165	2450	+35·5
15	66	501	1231	2339	2659	+27·0
16	62	454	1028	1992	2306	+17·0
17	64	770	1907	3814	4281	+ 6·0
18	171	1276	3238	7048	8146	− 6·5
19	244	2234	5317	10224	12314	−19·5
20	158	1091	2868	5669	6364	−32·5
21	193	1293	2671	4376	4549	−45·5
22	52	465	1144	1954	2051	−59·0
23	59	549	1087	1814	1954	−72·0

are given in the column N_{30}, and so on ; the sixth column, N_8 or All, gives the total number measured. As the hours of R.A. contain 7 and 8 plates alternately, the actual counts have been multiplied by factors to make them equivalent to ten plates (or forty square degrees) in each hour.

4. The magnitudes have been determined by the same methods adopted for the previous zones of the Hyderabad section. The numbers of stars in the whole circuit under the different headings in Table I. are added together and the magnitudes corresponding to the several diameters are deduced on the Chapman and Melotte scale from the table in the *Monthly Notices*, 75, 145. These are shown in Table II., together with the magnitudes computed from the formula $16{\cdot}10 - 0{\cdot}91\sqrt{d}$, which was determined independently for this zone. The coefficient of $\sqrt{d}$ in the above formula is nearly the same for the zones $-17°$, $-18°$, and $-19°$; the constant term is slightly larger, probably due to the longer exposures that had sometimes to be given, judging from the quality of the plates.

TABLE II.

Diameter.	50.	30.	20.	12.	8.
C. and M. Mag.	9·23	11·36	12·32	13·08	13·25
Calculated	9·61	11·12	12·04	12·95	13·53
Difference	−·38	+·24	+·28	+·13	−·28

5. It may be interesting to deduce the amount of galactic condensation from the data of Table I. Forming the log ratios of the numbers in the column N_{30} to those of the other columns and dividing them into two groups according to their galactic latitude, we have the following table :—

TABLE III.

Amount of Galactic Condensation of the Whole Zone.

Mean Gal. Lat.	No. in Group.	9·23.	12·32.	13·08.	13·25.
±54°	12	0	−3	−6	−6
±18°	12	+1	+3	+6	+6

From these we obtain the coefficient of condensation per magnitude per degree equivalent to −·00104.

6. A value of the galactic condensation may also be derived by considering the ratios of star density per square degree in the hours of R.A. whose galactic latitudes are numerically less than 20° to the corresponding density in the hours with latitudes greater than 40°. The following table gives the values of the galactic condensation for each of the magnitudes counts for which are available (the counts in R.A. 21^h seem to show abnormal results, and have been excluded).

TABLE IV.

Mag.	Gal. Condensation.
9·23	2·1
11·36	2·8
12·32	3·3
13·08	3·7
13·25	3·7

These figures are in fair accordance with the determinations of Professor Kapteyn from far more abundant material ranging over the whole sky (*Groningen Publications*, No. 18).

7. In what follows the data in Table I. have been treated exactly according to the method adopted by Professor Turner in several of his papers (see *M.N.*, **78**, 54). First the logarithms of the star densities per square degree ($\log N_\gamma$) are formed; an approximate value of the galactic condensation equal to ·0025 per degree per unit of log N is then applied, and the results cleared of the galactic effect. Thus $\log N_0$ is deduced from the formula $\log N_0 = (1 + \cdot 0025\gamma) \log N_\gamma + K_\gamma$. The mean values of $\log N_0$ for each column are given in Table V.; the values of the same quantity for the other Hyderabad zones are collected and given below for comparison.

TABLE V.

Mean Values of log N_0.

(Value of *d*.)

	50.	40.	30.	20.	12.	8.
−17°	0·68	...	1·46	1·79	2·08	2·31
−18	0·70	...	1·51	1·81	2·08	2·32
−19	0·56	0·96	1·49	1·83	2·12	2·27
−20	0·70	...	1·68	2·06	2·33	2·40

8 The residuals from these mean values of $\log N_0$ are shown in Table VI. (unit 0·01 in $\log N_0$).

TABLE VI.

Observed Residuals after correcting for Galactic Latitude for Hyderabad Zone −20°.

d / R.A.	50.	30.	20.	12.	8.	d / R.A.	50.	30.	20.	12.	8.
h						h					
0	+12	+ 2	+ 1	− 2	− 2	12	+20	+ 8	+ 3	+ 1	+ 1
1	+16	+ 9	+ 4	0	− 2	13	−41	−14	− 9	− 8	− 9
2	+ 1	−10	−16	−14	−10	14	−20	−10	−10	−10	−11
3	−21	− 3	− 9	−12	− 9	15	−23	−27	−23	−20	−21
4	−14	− 8	−10	− 7	− 5	16	−38	−46	−47	−44	−45
5	+ 1	−11	− 6	− 3	− 1	17	−47	−36	−34	−30	−32
6	−10	− 2	− 1	− 2	− 2	18	− 4	−13	−10	− 3	− 3
7	− 4	+13	+16	+10	+ 7	19	+28	+31	+33	+36	+37
8	+ 1	−11	−13	− 3	− 6	20	+24	+17	+25	+30	+28
9	+34	+16	+15	+18	+17	21	+46	+40	+37	+34	+29
10	+39	+29	+21	+19	+18	22	−13	− 2	+ 5	+ 5	0
11	+28	+23	+18	+10	+ 9	23	− 6	+10	+ 7	+ 6	+ 3

9. These columns are next submitted to harmonic analysis and the results are given in Table VII. for the different values of log N_0.

TABLE VII.

Harmonic Analysis of the Residuals.

log N_0.	Sin θ.	Cos θ.	Sin 2θ.	Cos 2θ.	Sin 3θ.	Cos 3θ.	Sin 4θ.	Cos 4θ.
0·70	+ ·064	+ ·027	− ·213	+ ·140	+ ·063	− ·102	− ·004	− ·011
1·68	+ ·054	+ ·041	− ·182	+ ·089	− ·005	− ·106	+ ·019	− ·026
2·06	+ ·028	+ ·043	− ·187	+ ·098	− ·030	− ·100	+ ·030	− ·026
2·33	+ ·014	+ ·042	− ·191	+ ·067	− ·022	− ·092	+ ·035	− ·033
2·40	+ ·014	+ ·042	− ·175	+ ·060	− ·015	− ·104	+ ·036	− ·030
1·07	− ·040	+ ·008	+ ·014	− ·039	− ·051	+ ·005	+ ·026	− ·012
1·00	− ·037	+ ·007	+ ·013	− ·036	− ·048	+ ·005	+ ·024	− ·011

The mean of the last three rows is subtracted from the mean of the first two, and the differences are given below the horizontal line at the foot of the table. In the next row the results are reduced to unit of log N_0 by dividing by 1·07.

10. From the above table we select the first harmonic,

Sin θ.	Cos θ.	R.	θ.
− ·037	+ ·007	·038	281°

The value θ_0 obtained from the equation of the spiral

$$\theta_0 = 247^\circ - 3{\cdot}66\delta \quad \text{is} \quad 247^\circ + 73^\circ = 320^\circ.$$

The difference $\theta - \theta_0 = -39^\circ$.

The results in Table VII. agree fairly well with those of the neighbouring zones collected and discussed by Professor Turner (*M.N.*, **81**, 531). The first harmonic may be written in the form

$$\cdot 038 \cos(\theta - 101^\circ).$$

It may be noticed that the very rapid change in the phase near -22° there suggested is well confirmed by the results of the present zone.

Nizamiah Observatory,
Hyderabad, Deccan, India:
1921 *Sept.*

Micrometrical Measures of Double Stars (*16th Series*).
By Rev. T. E. Espin.

Name.	R.A. 1900.	Decl.	P.	D.	Mags.	Nts.	Date 1921.
	h m	° ′	°	″			
h 1083	1 29·0	+60 47	166·6	4·70	10·9 11·4	2	·891 BC
			219·5	15·28	A=9·4	2	·891 AB
			277·6	9·89	9·6 11·8	3	·910 FG
			215·2	28·61		3	·910 AF
			332·8	5·47	12·2 12·7	2	·934 DE
			132·1	66·72		1	·947 AD
			36·4	14·15	*a*=11·2	2	·934 Aa
			84·6	24·55	*b* 11·9	2	·934 Ab
h 1108?	2 4·1	65 1	237·6	8·86	9·5 9·6	2	·989
Leonard 7	2 36·1	60 56	30·5	2·19	9·5 10·2	3	·156 AB
			20·2	16·22	*c*=13·0	3	·156 AC
Hu. 1063	3 30·1	62 27	335·2	2·92	9·0 9·2	5	·134
Σ 540	4 21·7	63 12	179·7	3·64	9·0 9·5	2	·170
Hu. 1085	26·1	62 40	204·2	3·87	8·2 11·0	1	·109
Espin 171	5 20·3	62 37	233·0	2·89	8·9 9·6	2	·134
Σ 1039	7 9·3	63 43	206·2	3·18	9·0 9·2	2	·272
Σ 1075	18·5	63 12	348·1	7·49	7·0 8·5	2	·272
Kr. 31	8 40·8	63 34	275·8	7·16	9·3 9·5	2	·211
Kr. 32	43·5	63 30	11·3	4·89	9·1 9·2	3	·226
Σ 2054	16 22·5	61 56	359·3	1·06	6·0 7·5	3	·681
Σ 2218	17 39·7	63 44	337·7	1·77	8·0 8·6	2	·677
Kr. 47	51·8	64 15	203·1	7·41	8·6 9·1	2	·738
Σ 2273	58·7	64 8	341·2	18·90	*b*=14·0	3	·670 Ab (new)
			284·1	20·91	7·0 7·3	2	·675 AB
			266·4	23·70	*c*=12·0	2	·675 AC
Σ 2332 rej.	18 25·2	64 52	260·8	10·56	9·2 10·7	2	·697
Σ 2343	29·3	65 2	213·8	8·72	8·5 8·9	2	·697 AB
Σ 2357	32·2	63 38	267·8	4·41	8·1 8·4	2	·683
Σ 2365 rej.	34·5	63 37	25·2	19·64	8·0 10·0	2	·683
Σ 2377 rej.	37·8	63 26	342·5	16·47	7·3 8·9	2	·784
Espin 126	38·7	63 42	26·5	4·37	9·8 10·2	2	·623 BC
Σ 2403	43·1	60 57	266·9	1·30	6·5 9·0	3	·706
h 2846	48·5	62 27	242·0	14·38	9·8 11·0	3	·731
Espin 191	19 5·4	61 7	239·7	6·12	9·2 9·7	3	·698
Σ 2509	15·9	63 1	336·1	1·00	7·0 8·0	2	·738
Σ 2564	36·0	63 35	167·2	9·80	8·9 9·5	3	·616

Name.	R.A. 1900.	Decl.	P.	D.	Mags.	Nts.	Date 1921.
	h m	° ′	°	″			
A.G.	19 53·5	+63 53	274·6	21·55	9·2 9·4	3	·670
Σ 2685	20 21·5	63 52	349·3	4·91	8·4 8·6	2	·784
Hu. 960	21 9·2	65 24	39·8	3·47	9·0 10·4	2	·843
Espin 137	16·7	61 26	74·9	2·88	9·5 12·0	4	·897 BC
Σ 2798	21·5	64 30	145·8	6·85	8·1 8·8	2	·796
Espin 141	38·6	60 45	189·2	2·03		2	·891
Σ 2842	48·4	63 33	99·2	4·00	8·3 10·0	5	·803
Espin 145	54·1	62 13	203·4	3·04	9·0 9·1	2	·860 AB
			233·3	18·53	C=13·2	2	·860 AC
Σ 2950	22 47·4	61 10	301·2	1·88	5·5 6·5	4	·914
Espin 142	49·4	61 37	326·3	6·89	8·5 9·2	3	·030
Espin 143	49·6	61 36	45·8	6·74	8·2 12·0	3	·030
Espin 218	52·6	64 15	335·0	2·88	9·8 10·0	2	·847 BC
			297·6	18·84	A=9·6	2	·847 AB
Espin 220	23 16·4	61 52	74·6	5·16	11·0 11·2	3	·898 BC
			83·2	36·82	A=8·2	2	·997 AB
Espin 149	35·2	63 45	115·9	5·81	8·8 9·2	2	·997

Notes.

h 1083. No other measure of this group. There are four stars, all mag. 14, forming a rhombus PA. The S. one, at P. 275°·3 D. 22″·99 from A, is double, mags. 14, 15, P. 170°±D. 3″±. There are also three stars near *a*, one fN and two PN. There is a 9·7 mag. PDE, and another little wide pair still further P, and a little pair PFG. A is B.D. +60°·276.

h 1108. Van Biesbroeck could not find *h* 1108, and suspected that this pair, 1° N. of *h*'s place, was it, and inferred motion. The measures do not confirm this. It is curious that *h* 1024 and *h* 1108, which each have virtually the same elements and magnitudes, are both missing. I have so far failed to find any pair which corresponds with either of them.

Σ 2273. The small star, *b*, was added with the 24-in.

A.G. $19^h\ 53^m·5$. Not given by *β* in the General Catalogue.

Espin 145. C was added with the 24-in.

Espin 143. A distant *comes*, 13·0 mag., at P. 257°·6.

Espin 149. The angle seems to be slowly diminishing.

New Double Stars.

No.	B.D.	R.A. 1900.	Decl.	P.	D.	Mags.	Nts.	1921.
	°	h m	° ′	°	″			
1865	+61·14	0 7·6	+61 17	177·1	2″·97	9·7 9·8	3	·295 BC
				121·1	25·00	A=9·0	3	·295 AB
1866	+61·125	29·7	61 19	18·3	5·02	8·5 13·0	2	·983
1867	+60·100	40·7	60 59	271·4	6·66	9·0 9·2	2	·843
1868	+56·139	45·6	56 32	183·8	6·83	9·2 11·0	3	·077
1869	+60·146	56·4	61 3	139·5	6·67	8·6 9·5	2	·981

No.	B.D.	R.A. 1900.	Decl.	P.	D.	Mags.		Nts.	1921.
	°	h m	° ′	°	″				
1870	+60·204	1 12·6	+60 21	284·6	4·10	9·2	9·3	2	·850
1871		30·1	60 15	244·4	1·87	9·9	12·0	2	·847
1872	+60·361	43·9	60 35	349·2	3·14	10·5	10·6	2	·981 BC
				270·8	25·51	A=9·5		2	·981 AB
1873	+60·407	52·6	60 26	232·9	3·50	9·2	14·0	2	·981
1874		2 10·1	56 17	144·1	3·53	9·5	12·2	2	·072
1875	+60·544	33·4	60 42	161·9	1·85	9·1	9·8	4	·155
1876		40·0	60 43	297·7	3·57	10·0	10·2	4	·078
1877	+60·571	40·9	60 58	267·1	5·27	9·5	13·2	2	·158
1878	+60·639	3 4·1	60 34	80·9	5·37	9·0	9·6	2	·013
1879	+61·570	16·0	61 11	284·7	5·83	8·4	11·7	2	·013
1880	+62·586	29·0	62 26	36·4	4·16	9·4	13·5	4	·132
1881		31·4	62 50	335·6	3·00	10·5	10·6	2	·134
1882	+62·657	4 0·1	62 15	289·6	7·22	8·1	11·2	2	·046
1883	+63·495	14·2	63 7	234·5	3·47	9·0	10·8	2	·170
1884		27·0	63 8	291·4	2·08	10·0	10·2	2	·152
1885	+62·715	48·0	62 17	109·8	3·26	9·5	11·5	3	·067
1886	+63·589	5 23·2	63 54	54·1	6·60	9·3	11·5	2	·609
1887	+62·764	23·9	62 7	105·8	5·35	8·5	9·6	3	064
1888		40·4	62 20	63·6	3·89	10·0	11·2	2	·134
1889		6 7·1	62 5	22·6	2·93	11·0	11·4	4	·163
1890		22·1	62 57	120·7	5·43	9·9	10·7	3	·224
1891	+60·974	23·3	60 48	192·4	4·57	8·6	9·2	2	·041
1892	+60·1009	46·5	60 37	285·3	7·18	9·2	9·8	2	·119
1893	+63·676	51·4	63 39	154·1	5·93	9·2	11·2	2	·248
1894	+62·915	7 4·0	61 59	100·9	6·06	9·2	13·0	2	·140
1895	+62·933	22·5	62 43	284·5	9·33	7·1	9·4	2	·152
1896		42·5	64 40	23·3	2·96	9·7	10·8	2	·272
1897		45·2	62 28	241·1	3·15	10·5	11·1	3	·134
1898		46·5	62 29	235·4	2·79	10·0	10·1	2	·140
1899	+63·774	8 13·7	63 37	204·0	8·52	8·7	10·4	4	·283
1900	+63·780	19·3	63 27	66·7	2·96	9·5	13·2	3	·259
1901	+62·1056	9 5·5	62 6	75·2	6·43	9·5	13·2	3	·167
1902	+63·853	30·4	63 16	268·2	8·79	8·8	9·2	2	·248
1903	+62·1104	10 0·1	62 42	266·9	4·34	9·0	11·3	3	·209
1904	+64·773	6·4	64 0	17·3	6·37	9·0	12·5	3	·294
1905	+64·788	23·4	63 52	200·3	3·99	11·2	13·0	4	·302 BC
				65·6	47·27	A=7·0		2	·279 AB
1906		11 3·7	61 56	273·3	2·83	10·0	10·5	2	·279
1907		21·2	59 56	263·8	2·07	10·0	10·2	4	·294

No.	B.D.	R.A. 1900.	Decl.	P.	D.	Mags.		Nts.	1921.
	′ °	h m	° ′	°	″				
1908	+64·1178	17 10·5	+63 59	138·7	5·93	9·4	13·0	4	·665
1909		19·7	63 21	254·1	5·93	10·2	10·3	2	·591
1910	+65·1211	42·2	65 8	55·9	5·39	9·3	9·4	3	·721
1911	+66·1113	18 34·1	66 4	296·1	6·29	8·5	10·0	3	·819
1912		49·0	65 21	23·5	7·47	9·4	11·0	3	·836
1913	+64·1319	19 2·8	64 7	313·4	6·31	9·3	14·0	2	·623
1914		4·6	64 54	350·8	3·77	10·5	11·5	2	·680
1915		6·2	64 40	146·3	3·54	10·5	11·0	3	·673
1916		22·9	64 25	230·9	6·00	9·6	9·8	3	·614
1917	+63·1563	46·6	63 48	344·3	7·72	8·1	12·2	3	·660
1918	+63·1646	20 37·3	63 44	283·4	2·58	9·4	10·2	3	·670
1919	*h* 1656	21 26·0	64 59	5·2	6·18	8·6	14·0	2	·831 AB
				157·2	13·78	C=9·5		2	·831 AC
				13·5	14·69	D=8·9		3	·836 AD
	+63·1744	30·2	63 23	351·7	12·14	9·5	10·3	2	·796
	+63·1748	31·7	63 34	227·7	12·98	7·5	12·0	2	·796
1920		47·8	63 33	152·3	4·24	9·8	11·5	3	·792
1921		48·5	63 46	352·1	4·25	10·0	10·2	3	·792
1922		51·3	61 6	168·4	3·93	11·0	11·1	2	·820
1923	+60·2346	22 5·2	60 36	229·5	3·14	10·5	12·7	2	·813 BC
				341·3	25·01	A=8·2		3	·817 AB
1924	+60·2365	10·9	60 58	142·7	6·21	9·4	9·7	2	·838
1925	+61·2275	13·3	61 54	357·2	4·50	9·0	12·0	2	·847
1926		13·4	61 33	255·5	3·37	10·0	12·0	2	·854
1927	+60·2463	54·5	60 39	227·1	3·19	9·3	12·5	3	·825
1928	+60·2527	23 17·9	60 29	179·7	2·73	9·5	11·0	2	·813
1929	+60·2572	28·7	61 10	295·1	2·85	9·4	11·7	3	·852
1930		31·2	61 16	203·8	1·87	10·0	10·5	3	·878
1931	+60·2620	43·7	60 43	213·6	5·07	9·1	11·8	3	·573
1932	+61·2555	48·2	61 24	343·4	6·20	9·2	9·3	3	·573
1933	+60·2659	56·9	60 24	350·2	2·22	9·5	9·6	2	·824

Notes.

1868. A *comes* f.
1889. In field with B.D. +62°·827 at P. 23°·2.
1893. *Comes*, 14 mag. f.
1899. A is very much underrated in B.D., one or both variable ?
1923. Another 10 mag. at P. 17°·6, D. 25″±.

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The constant practice in the comparison of chronometers has given the Greenwich observers great facility and considerable accuracy in the reception of the ordinary signals. This is shown (1) by the agreement of observers *inter se*, (2) by the agreement of the ordinary with the rhythmic signals, (3) by the agreement of the signals received by ear with those recorded automatically.

The agreement of the observers *inter se* is sufficiently illustrated in the paper already referred to.

The difference between the rhythmic and ordinary signals found from 174 comparisons made on the same evenings by Mr. Bowyer between 1912 October 17 and 1914 March 31 in the sense rhythmic-observed was $+^s\cdot004$, and the mean discordance between the two was $\pm^s\cdot060$.

The comparison between the registered signals and the observations of Mr. Bowyer and Mr. Stevens for 1921 January to September are :—

	Mr. Bowyer.			Mr. Stevens.		
	No.	Diff.	Disc.	No.	Diff.	Disc.
		s	s		s	s
Paris 10^h	166	$+\cdot012$	$\pm\cdot054$	81	$+\cdot007$	$\pm\cdot058$
Paris $10^h\ 45^m$	185	$-\cdot006$	$\pm\cdot051$	158	$-\cdot006$	$\pm\cdot050$
Nauen 12^h	121	$+\cdot031$	$\pm\cdot068$	90	$+\cdot013$	$\pm\cdot058$

These comparisons show that the reception of the ordinary signals does not involve any large systematic error, and, in view of this, the eye and ear observations have been combined with the registration results, the latter being given double weight.

Comparison has been made of the automatic registers of the Paris $10^h\ 0^m$ and $10^h\ 45^m$ signals. From 173 observations a mean difference of $+^s\cdot011$ was found with a discordance between the two signals on any day of $\pm^s\cdot031$. This close agreement between the signals shows that the errors of the transmission and of the records by a syphon recorder are satisfactorily small. In the results given below the observations of the $10^h\ 0^m$ and $10^h\ 45^m$ signals have been combined.

An interesting comparison of the $10^h\ 45^m$ time signal with the rhythmic series at $10^h\ 0^m$ for the month of December is given below. Both of these sets of signals were registered automatically on a syphon recorder.

No.	B.D.	R.A. 1900.	Decl.	P.	D.	Mags.	Nts.	1921.
	°	h m	° ′	°	″			
218		19 40·1	+65 56	222·1	4·51	9·7 10·7	3	·721 AB
				102·7	5·91	10·7 13·0	2	·720 CD
				350·6	56·60		1	·728 AC
219		20 17·2	65 7	337·5	2·81	10·0 10·1	3	·795
220	+64·1450	31·7	64 21	89·1	6·95	8·6 9·5	2	·653
221	+65·1563	21 12·2	65 41	174·1	3·43	9·3 9·4	2	·843
222		17·4	64 49	265·2	1·34	10·1 10·3	4	·852
223		31·1	63 46	133·4	2·27	10·0 10·5	2	·850
224		34·1	64 22	112·1	5·16	10·0 12·0	3	·857
225		55·9	63 30	255·6	6·24	9·7 12·0	2	·796
226	+63·1806	22 3·6	63 17	112·9	2·60	9·3 9·5	2	·831
227	+64·1652	19·4	64 23	208·7	8·41	9·5 13·0	2	·934
228	+64·1653	20·4	64 22	215·1	7·68	8·6 12·0	2	·934
229	+64·1677	28·0	64 57	177·2	4·23	8·6 10·3	2	·891
230	+63·1862	29·1	63 36	103·9	8·06	9·3 13·0	2	·822
231		44·8	63 53	184·4	2·83	11·8 12·2	2	·850 BC
				4·4	40·31	A=9·8	2	·850 AB
232	+63·1900	50·0	63 52	224·5	6·53	8·9 12·7	4	·825
233	+64·1739	54·9	64 23	216·5	3·43	9·5 9·6	2	·843
234		23 18·8	64 20	101·8	2·06	10·0 10·4	2	·831
235	+62·2238	25·6	62 31	265·0	3·66	9·4 9·5	3	·823
236	+63·2066	47·0	64 4	152·8	4·77	9·2 10·5	2	·853
237	+64·1871	48·8	65 11	264·0	2·67	9·5 11·0	3	·919
238	+64·1879	51·4	64 45	256·7	2·08	9·8 11·1	3	·880
239		53·8	62 44	196·5	3·22	10·7 11·2	3	·894
240	+62·2344	56·0	63 3	41·7	6·24	9·2 10·6	3	·851
241	+62·2349	56·6	62 36	304·3	2·82	9·0 11·2	5	·881

Notes.

184. Van Biesbroeck, P. 64°·5, D. 9″·27, 3 nts. 1909·041 for AC. It is strange that the duplicity of B escaped him, if so easy then, as now.
209. Also *comes*, 13·5 mag. P. 77°·5, D. 34″·7, single setting.
218. Also *comes*, 14·0 mag. between the pairs, rather nearer CD.

Stars faint or Missing.

	h m s	° ′	
B.D.+60°15	0 8 8·6	+60 4·1	mag. 9·5 (1855) 1921 Nov. 12, Dec. 29, 11·0 mag.
B.D.+60·674	3 11 52·4	+60 55·5	mag. 9·5 (1855) 1921 Jan. 6, no star here
B.D.+65·1462	20 24 27	+65 2·0	mag. 9·5 (1855) Milburn, Decl. +65° 4′·3
B.D.+62·2200	23 8 11	+62 48·2	mag. 9·5 (1855) Milburn, Decl. +62° 46′·2

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The constant practice in the comparison of chronometers has given the Greenwich observers great facility and considerable accuracy in the reception of the ordinary signals. This is shown (1) by the agreement of observers *inter se*, (2) by the agreement of the ordinary with the rhythmic signals, (3) by the agreement of the signals received by ear with those recorded automatically.

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		s	s		s	s
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These comparisons show that the reception of the ordinary signals does not involve any large systematic error, and, in view of this, the eye and ear observations have been combined with the registration results, the latter being given double weight.

Comparison has been made of the automatic registers of the Paris $10^h\ 0^m$ and $10^h\ 45^m$ signals. From 173 observations a mean difference of $+^s\cdot011$ was found with a discordance between the two signals on any day of $\pm^s\cdot031$. This close agreement between the signals shows that the errors of the transmission and of the records by a syphon recorder are satisfactorily small. In the results given below the observations of the $10^h\ 0^m$ and $10^h\ 45^m$ signals have been combined.

An interesting comparison of the $10^h\ 45^m$ time signal with the rhythmic series at $10^h\ 0^m$ for the month of December is given below. Both of these sets of signals were registered automatically on a syphon recorder.

	Rhythmic Signal.	Time Signal.	Diff.			Rhythmic Signal.	Time Signal.	Diff.	
	s	s		s		s	s		s
Dec. 1	59·97	59·93	+	0·04	Dec. 15	0·09	0·07	+	0·02
2	·89	·87	+	2	16	·08	·11	−	3
3	·81	·79	+	2	17	·04	·01	+	3
5	·94	·93	+	1	19	59·89	59·89		0
6	·93	·93		0	20	0·00	0·00		0
7	·95	·93	+	2	21	·09	·11	−	2
8	0·06	0·05	+	1	22	·10	·07	+	3
9	·12	·11	+	1	23	·14	·12	+	2
10	·15	·16	−	1	24	·08	·08		0
12	·23	·21	+	2	29	·14	·14		0
13	·20	·20		0	30	·12	·12		0
14	·07	·05	+	2	31	·10	·11	−	1

(The signal times given result from the use of provisional clock errors. Any alteration will affect both sets of signals to the same extent.)

The accordance of the time signals with the rhythmic signals is remarkably close, especially when it is remembered that the $10^h\ 45^m$ time signal sent from the Paris Observatory involves the setting of a mean time clock by the sidereal clock on which the rhythmic signals have been observed and the emission of a signal by a high-power wireless station at a prearranged time.

The means for the separate months are given in the following tables for Paris from 1912 August to 1921 October; and for Paris, Nauen, and Annapolis from 1920 January to 1921 October.

Paris W/T Signals. Means for Each Month.

	1912.	1913.	1914.	1915.	1916.	1917.	1918.	1919.
	s	s	s	s	s	s	s	s
Jan.	...	0·02	9·90	0·06	0·00	0·01	0·01	9·92
Feb.	...	0·05	9·99	0·02	9·92	0·02	0·03	0·05
March	...	0·04	0·06	0·05	9·98	0·03	0·11	9·98
April	...	0·11	9·99	0·09	0·01	0·05	0·10	0·05
May	...	0·14	9·98	0·04	0·01	9·98	0·05	0·07
June	...	9·98	9·94	0·05	9·96	0·00	0·04	0·18
July	...	9·88	9·97	0·07	9·96	9·95	9·98	0·06
Aug.	0·04	9·89	0·00	0·06	9·92	9·98	0·00	0·11
Sept.	0·22	9·96	0·05	9·98	9·98	0·00	9·98	0·04
Oct.	0·01	0·01	0·05	0·03	9·90	0·06	0·10	0·16
Nov.	0·09	0·10	0·11	0·01	9·92	0·06	0·00	0·15
Dec.	0·16	9·96	0·12	9·96	9·92	0·06	0·01	0·07
Mean	0·11	0·01	0·01	0·03	9·95	0·02	0·03	0·07

W/T Signals. Means for Each Month.

	1920.			1921.		
	Paris.	Nauen.	Annapolis.	Paris.	Nauen.	Annapolis.
	s	s	s	s	s	s
Jan.	0·14	9·98	0·18	0·24	0·12	0·06
Feb.	0·14	9·95	0·12	0·11	0·08	0·02
March	0·09	9·93	0·13	0·17	9·97	0·24
April	0·06	0·05	0·12	0·10	9·96	0·21
May	0·12	0·08	0·17	0·10	0·00	0·21
June	0·12	0·04	0·13	0·10	0·00	0·13
July	0·10	0·08	0·18	0·21	0·06	0·16
Aug.	0·15	0·04	0·18	0·13	0·06	0·15
Sept.	0·08	0·04	0·13	0·03	0·16	0·18
Oct.	0·10	9·99	0·07	0·07	0·00	0·18
Nov.	0·12	9·99	0·04	0·04	9·94	0·18
Dec.	0·21	0·06	0·04	0·09	0·01	0·15
Mean	0·12	0·02	0·12	0·12	0·03	0·16

With reference to these figures, it may be noted that it is stated that there is a lag of $0^{s}\cdot07$ on the land line between Washington and Annapolis, and, further, the time of transmission is approximately $0^{s}\cdot02$, so that $0^{s}\cdot09$ must be subtracted from the Annapolis figures to make them comparable with those of Paris and Nauen.

If the positive value is regarded as due to error in longitude and not as arising from instrumental errors either in time determination, sending, or reception of signals, these figures imply that Greenwich is about $0^{s}\cdot03$ to westward of its assumed position relatively to Berlin, Paris, and Washington.

There are two features in the tables given above which attract notice. The first is the increase in the recorded times of the Paris signals, which commenced about the middle of 1919 and has continued to the end of 1921 September. We have no earlier observations of the Nauen and Annapolis signals to indicate to what extent this change depends on Greenwich and to what extent on Paris.

The other feature is the somewhat abrupt changes which sometimes occur between the means for successive months. As has been pointed out by Professor Sampson, and as we have noticed at Greenwich, the times of reception of the time signals generally agree closely for successive days, but are systematically large or small over extended periods. This is shown in the following table, where means are taken for each week, Monday to Saturday, from the beginning of 1920.

W/T Signals. Means for Each Week.

Week commencing	Paris.	Nauen.	Annapolis.
	s	s	s
1920 Jan. 5	0·14	0·01	0·22
12	·16	·09	·23
19	·12	9·96	·15
26	·17	·90	·13
Feb. 2	·14	·98	·16
9	·28	·99	
16	·08	·91	
23	·05	·92	
Mar. 1	·10	·95	·13
8	·12	·92	·14
15	·10	0·03	·12
22	·07	·03	·16
29	·06	·15	·14
Apr. 5	·07	·10	·13
12	·13	·00	·12
19	·05	·04	·15
26	·03	·00	·04
May 3	·16	·18	·18
10	·11	·00	·13
17	·11	·09	·21
24	·10	·12	·16
31	·13	·13	·24
June 7	·19	·12	·13
14	·10	·01	·02
21	·06	9·98	·13
28	·23	0·04	·24
July 5	·10	·15	·22
12	·23	·14	·19
19	·05	9·99	·21
26	9·92	·96	·14
Aug. 2	0·12	0·13	·24
9	·16	·03	·16
16	·05	9·94	·05
23	·21	0·11	·22
30	·05	·05	·18
Sept. 6	9·92	·06	·14
13	0·21	·04	·11
20	·10	·00	·06
1920 Sept. 27	0·12	0·07	0·10
Oct. 4	·19	·09	·19
11	·03	9·91	·20
18	·04	·93	·01
25	·03	·96	·03
Nov. 1	·02	·91	·10
8	·10	·97	·10
15	·16	0·05	
22	·20	·00	9·98
29	9·96	9·94	9·87
Dec. 6	0·14	0·04	9·95
13	·25	·02	0·12
20	·26	·01	·14
27	·43	·25	·07
1921 Jan. 3	·38	·18	·04
10	·27	·17	·00
17	·18	·05	·09
24	·10	·11	·08
31	·08	·09	9·95
Feb. 7	·12	·16	0·05
14	·10	·08	·06
21	·11	9·98	
28	·20	0·09	
Mar. 7	·24	·03	
14	·24	·01	·21
21	·09	9·86	·23
28	·02	·92	·28
Apr. 4	·01	·93	·16
11	·07	·98	·23
18	·17	·99	·25
25	·19	·97	·25
May 2	·09	·97	·24
9	·09	0·07	·20
16	·20	·06	·26
23	·07	9·94	·17
30	·06	·97	·07
June 6	·08	0·06	·17
13	·08	9·98	·14

W/T Signals—continued.

Week commencing	Paris.	Nauen.	Annapolis.	Week commencing	Paris.	Nauen.	Annapolis.
	s	s	s		s	s	s
1921 June 20	0·21	0·01	0·12	1921 Aug. 15	0·09	0·07	0·17
27	·00	9·96	·11	22	·00	·01	·13
July 4	·02	0·01	·16	29	·04	·09	·19
11	·26	·10	·12	Sept. 5	·04	·16	·20
18	·35	·04	·13	12	·07	·14	·20
25	·32	·13	·19	19	·01	·12	·19
Aug. 1	·34	·10	·25	26	·00	·09	·07
8	·19	·02	·17				

These fluctuations appear to depend on the time determinations at the four observatories, and are not due to errors in the transmission or reception of the signals. If the means be taken for all weeks in which signals have been received from Paris, Nauen, and Annapolis, and a general mean for the whole period taken out, the mean discordance is $\pm 0^{s}\cdot05$. The weeks in which the time determination at Greenwich differs by $0^{s}\cdot10$ or more from the mean of the three sets of signals are:—

	s	
1920 Oct. 18–Oct. 23	−0·11	(1)
1920 Nov. 29–Dec. 4	−0·18	(2)
1920 Dec. 27–1921 Jan. 1	+0·15	(3)
1921 Jan. 3–Jan. 8	+0·10	(4)
1921 July 25–July 30	+0·11	(5)
1921 Aug. 1–Aug. 6	+0·13	(6)

These periods have been examined in detail. (1) For the period Oct. 18–Oct. 23 determinations of clock error were obtained every day as follows:—

Civil Time.	Error of Clock.	No. of Stars.	C.E. adopted at 11^{h}.
d h	s		s
Oct. 18 11	2·00	5	+2·00
19 18	1·87	6	+1·91
20 17	1·72	16	+1·76
21 19	1·55	7	+1·60
22 20	1·43	2	+1·48
23 21	1·26	8	+1·32

(2) For the period Nov. 29–Dec. 4 the determinations of clock error and the adopted errors at 11^h are :—

Civil Time.		Error of Clock.	No. of Stars.	C.E. adopted at 11^h.
d	h	s		s
Nov. 26	20	−0·96	4	
29				−1·18
30	11	−1·25	1	−1·25
Dec. 1	11	−1·27	9	−1·27
2				−1·34
3	22	−1·44	5	−1·40
4				−1·49
9	18	−1·99	11	

(3) and (4) For the period 1920 Dec. 27–1921 Jan. 9 :—

Civil Time.		Error of Clock.	No. of Stars.	C.E. adopted at 11^h.
d	h	s		s
Dec. 26	4	−1·23	8	
27				−1·06
28	12	−0·92	2	−0·93
29	0	−0·87	11	−0·81
30	16	−0·65	5	−0·68
31				−0·56
Jan. 1	6	−0·45	5	−0·43
2				
3				−0·28
4				−0·21
5	19	−0·12	2	−0·14
6				−0·10
7				−0·07
8	0	−0·04	5	0·00
9				
10	20	+0·23	5	

(5) and (6) For the period 1921 July 25–Aug. 7 :—

Civil Time.		Error of Clock.	No. of Stars.	C.E. adopted at 11^h.
d	h	s		s
July 25	21	+16·20	6	+16·38
26	21	+15·81	3	+15·96
27				+15·62
28				+15·32
29				+15·00
30	9	+14·69	1	+14·67
31				
Aug. 1				
2	22	+13·49	8	+13·65
3	19	+13·17	6	+13·27
4	10	+12·97	1	+12·96
5				+12·70
6	17	+12·38	9	+12·44

It will be seen that in the first of these periods the determinations of clock error at Greenwich were very good, but in the other periods, owing to the weather, there are few observations. It is, however, difficult to see how errors of the required amount can arise from this cause. The determinations of azimuth have been examined, and are quite satisfactory, except for the second period. In this case there are only two determinations. They are in good agreement, but differ by $1''\cdot5$ from determinations of previous and successive weeks. Further examination of all available stars shows that the discordance $-0^s\cdot18$ is explained to the extent of $0^s\cdot10$ by the poverty of azimuth determinations.

It should be pointed out that the Greenwich Clock Errors are those finally adopted. Necessarily, the W/T signals have been sent out with a clock error which is extrapolated. It is likely that where, owing to bad weather, a clock rate has had to be carried forward several days, the discordances may be diminished by the adoption of clock errors derived in the light of subsequent observations.

Wireless Time Service of the Pulkovo Observatory.

(*Communicated by the Astronomer Royal.*)

Details of the W/T time-signals sent by the stations of Petrograd and Moscow have been received from M. N. Dnieprovsky, Chief of the Time Service of the Pulkovo Observatory. As this information may be of service to astronomers and others interested in W/T time-signals, the substance of M. Dnieprovsky's communication and a few observations taken at Greenwich are embodied in this note.

In the first place, it appears that standard time has been adopted in Russia, the time being 2 hours in advance of Greenwich. The time-determination is made at Pulkovo Observatory, which is connected by land line with the W/T stations of Petrograd and Moscow. The programme of signals sent out is as follows, all times being expressed in Greenwich Mean Time (Civil), *i.e.* reckoning from midnight:—

Petrograd. G.M.T. (Civil).	Nature of Signal.	Moscow. G.M.T. (Civil).
h m s – h m s		h m s – h m s
	Call Signals.	
19 0 0 – 19 2 0	Series of v ··· –	21 55 0 – 21 57 0
	Warning Signals.	
19 2 20 – 19 2 50	Series of t –	21 57 20 – 21 57 50
19 3 20 – 19 3 50	Series of m – –	21 58 20 – 21 58 50
19 4 20 – 19 4 50	Series of o – – –	21 59 20 – 21 59 50

Time-Signals.

Consisting of three dashes each of one second duration, separated by one second space, commencing

h	m	s	h	m	s
19	3	0	21	58	0
19	4	0	21	59	0
19	5	0	22	0	0

Correction to times of signal given at

h	m	s	h	m	s
19	5	10	22	0	20

Rhythmic Signals.

h	m	s		h	m	s
19	6	0	–	19	13	49

Sixteen series of 22 dots: two series every minute from 0^s–19^s and 30^s–49^s. The spacing of the dots is 66 in 60 seconds of time, or 11 in 10 seconds.

Six series of 31 signals commencing at $22^h\ 2^m\ 30^s$, each set being separated by the omission of 9 signals. This is followed by a set of 121 signals. This is followed by six series of 31 signals similar to the previous set. The interval between contiguous signals is 48/49 of 1 second.

The time of the first rhythmic signal is not communicated by wireless. In order to use interpolated instead of extrapolated clock errors, the time will be published at intervals.

The Moscow time-signals have been observed on a few occasions at Greenwich, but it was not understood when they were first taken that the figure telegraphed after the signals gave a correction to the time-signal. The correction is shown on January 7 and January 9.

Date.	App. Error of Signal.	Date.	App. Error of Signal.
1921	s	1922	s.
Dec. 16	0·02 fast	Jan. 2	0·57 slow
21	·46 slow	4	·28 ,,
23	·77 ,,	5	·43 ,,
24	·41 ,,	7	$·43 - {}^s·10 = {}^s·33$ slow.
28	·45 ,,	9	·60 − ·30 = ·30 ,,
29	·44 ,,		
30	·67 ,,		
31	·46 ,,		

Note sur la Détermination de l'Heure.

Par H. Philippot et F. Moreau.

(*Communicated by Professor R. A. Sampson.*)

Généralités.—Déterminer l'heure à un moment donné c'est évaluer la correction qu'il faut appliquer à l'heure marquée, à ce moment, par une pendule pour obtenir l'heure exacte. Les observations astronomiques qui permettent de déterminer l'heure devinrent *plus précises* à la fin du siècle dernier quand on substitua à la méthode de l'œil et de l'oreille, l'inscription chronographique et le micromètre enregistreur ; elles devinrent *plus commodes* dans ces dernières années par l'utilisation du chronographe imprimant et du micromètre enregistreur à entraînement automatique du fil.* Mais jusqu'au début de ce siècle les pendules soumises aux variations de température et de pression gardaient mal le temps et l'astronome ne pouvait garantir son heure à un dixième de seconde près que pour le moment de l'observation astronomique. Malgré les efforts ingénieux des horlogers qui cherchèrent à compenser les effets des variations de température et de pression (balanciers compensateurs Graham, Harrison, Leroy, Airy, et Rédier) les marches des pendules étaient troublées par ces variations et, d'autre part, il était bien difficile de déterminer les coefficients suivant lesquels les marches se modifiaient. L'influence des changements de température est très complexe ; aucune formule, si savante soit-elle, ne peut en donner une expression convenable : la température, en effet, ne modifie pas seulement la longueur du balancier mais elle modifie aussi les frottements, la densité de l'air et son humidité et remarquons encore qu'en raison de la différence de conductibilité des diverses parties de l'horloge, toute l'influence d'un changement de température ne se fait pas immédiatement sentir.

L'effet de la variation de la pression atmosphérique (variation de la densité de l'air) se détermine mieux et l'on admet généralement qu'une pendule retarde journellement de $0^{s}\cdot018$ pour une augmentation de pression de $1^{mm}\cdot$ de mercure. Mais nous avons parfois remarqué que les pendules n'obéissent pas immédiatement au changement de pression et il semble que la variation de marche dépende de la forme du balancier.

Pour toutes ces raisons, la marche d'une pendule soumise aux variations de température et de pression est toujours incertaine et l'expérience a montré que, même lorsqu'on dispose de plusieurs pendules, l'heure extrapolée peut être erronée d'une seconde au bout d'une dizaine de jours.

Un grand progrès a été réalisé dans ces dernières années par l'utilisation des pendules à température et à pression constantes.† Ces

* J. C. Hammond, "Personal Errors in the Use of a Self-registering Micrometer," *Ast. Jour.*, No. 787 (1921).

† Professor Sampson, "On Clock Errors," *M.N.R.A.S.*, 86, 89 (1920). G. Bigourdan, "Note sur les corrections des sign. horaires," *Bul. Ast.*, t. i. fasc. 5 (1921). Dr. S. Riefler, *Präzisions-Pendeluhren . . . München* (1907). C. Howe, "The Rate of the Riefler Sidereal Clock, No. 56," *Ast. Jour.*, No. 524 (1902). M. L. Zimmer, "The New Subterranean Clock-room . . . ," *Ast. Jour.*, No. 762 (1920).

pendules sont placées dans une cave où régne une température constante; elles sont enfermées dans des cloches hermétiques qui contiennent de l'air—ou un gaz (inexplosible)—à une pression inférieure aux plus faibles pressions atmosphériques constatées.

L'observatoire royal de Belgique possède quatre pendules Riefler sous cloche. Ces horloges ont des marches diurnes inférieures à $0^{s}\cdot04$ et des écarts moyens de la marche diurne (moyenne des différences entre la marche moyenne et la marche de chaque jour) voisins de $0^{s}\cdot01$.

Chaque jour à la même heure, les pendules sont comparées à la pendule principale et l'on fait concourir ainsi toutes les horloges à la détermination de l'état de l'une d'elles. Comme les marches sont faibles et très régulières, l'heure extrapolée, calculée d'après nos quatre meilleurs garde-temps, est rarement erronée de plus de $0^{s}\cdot1$ au bout de quatre ou cinq jours. Cette bonne conservation de l'heure n'est pas seulement précieuse aux périodes de mauvais temps pendant lesquelles les observations astronomiques sont impossibles, mais il est d'un grand intérêt, nous semble-t-il, de comparer l'heure probable déduite des marches des pendules—heure extrapolée—à l'heure obtenue par une observation astronomique. Tous les astronomes qui s'occupent de la détermination de l'heure sont parfois surpris, en examinant les graphiques qui représentent les marches des pendules, de voir que tous leurs garde-temps sont perturbés dans le même sens et d'une même quantité. Quand les pendules des observatoires étaient soumises aux variations de température et de pression, on admettait souvent que ces perturbations étaient dues aux changements de pression et de température dont on ne mesurait pas assez parfaitement l'influence. Les pendules sous cloche et à température constante sont maintenant débarassées de cette influence et les heures extrapolées de chacune d'elles concordent souvent si bien que l'on s'étonne parfois de leur désaccord avec l'heure déterminée astronomiquement. Se sentant si bien contrôlé par les pendules, l'astronome va se voir obligé d'être plus minutieux dans le choix de ses étoiles et dans la détermination de ses corrections instrumentales, de perfectionner, de changer, peut-être, sa méthode de détermination de l'heure. Sans doute l'étude attentive de ces désaccords conduira-t-elle à d'intéressantes recherches. M. Zimmer, de l'observatoire de Cordoba (Argentine) a déjà signalé que les corrections des pendules déterminées par des observations faites après le coucher du soleil sont plus grandes de $0^{s}\cdot04$ ou $0^{s}\cdot05$ que celles déterminées par des observations du matin.* La même constatation a été faite par M. Tucker de l'observatoire Lick. Si ce fait curieux ne peut s'expliquer par une variation de l'équation personnelle, il serait intéressant d'en rechercher la cause. Il semble bien, comme l'a déjà fait remarquer

H. Kienle, "Untersuchungen über Pendeluhren," *Neue ann. der K. Stern. München*, Band v. Heft 2 (1918). H. Philippot et E. Delporte, *Description des installations du service de l'heure de l'observatoire royal de Belgique* (1912). H. Philippot et F. Moreau, "Le service de l'heure à l'observatoire d'Uccle," *Rev. horlogère de Belg.*, mars et avril 1921.

* M. L. Zimmer, "On an Annual Term in the Right Ascensions," *Ast. Jour.*, No. 745 (1919), No. 766 (1920), No. 792 (1921). R. H. Tucker, "Comparison of Observed Clock Corrections at Sunset and Sunrise," *Lick Obs. Bul.*, No. 330 (1921).

M. Stewart,* que la réfraction latérale † ne puisse expliquer ce phénomène puisque son influence varierait avec la déclinaison de l'étoile et que l'observation ne confirme pas cette variation.

S'agirait-il ici d'un changement de méridien qui entraînerait un changement d'heure et même de longitude? Rappelons-nous que le méridien d'un lieu est le plan déterminé par la verticale et par une droite menée, par ce lieu, parallèlement à l'axe de rotation de la terre. Or on sait que cet axe n'est pas invariablement lié à la terre (variation des latitudes) et que la direction de la verticale est troublée par l'attraction luni-solaire.‡ Il n'est donc pas impossible que le méridien éprouve une certaine oscillation. On objectera que les plus minutieuses mesures de l'inclinaison de l'axe de la lunette méridienne n'accusent pas de variations à courte période, mais remarquons que le sol n'est pas d'une rigidité absolue § et qu'il subit—ainsi que l'axe de rotation de la lunette—des mouvements qui dépendent de la variation de la verticale (marées de l'écorce terrestre). Un changement d'inclinaison de la lunette peut avoir deux causes: (1) une déviation de la verticale, (2) un mouvement du sol (et des piliers); il est malheureusement difficile de dire la part qu'il faut attribuer à chacune d'elles et si le sol se déformait docilement, la verticale pourrait être déviée sans qu'il s'ensuive pour cela une variation de l'inclinaison de la lunette. Mais objectera-t-on peut-être encore, le calcul montre || que la déviation de la verticale dans le sens du parallèle, déviation produite par l'action luni-solaire, ne peut être que de quelques millièmes de seconde d'arc, si petite qu'elle ne pourrait être révélée par nos observations astronomiques. Il s'agit là de l'action luni-solaire directe; à celle-ci s'ajoute, sans doute, la déviation due aux marées de l'océan et au déplacement de masses au sein de la terre.

Pour voir si le phénomène révélé par M. Zimmer peut s'expliquer par un changement de méridien il conviendrait, nous semble-t-il, de l'étudier en des points de latitudes très différentes. Dans le voisinage des pôles, en effet, les deux droites qui définissent le méridien forment un très petit angle et un petit changement dans l'orientation de l'une d'elles pourrait modifier sensiblement la position de ce plan fondamental et changer, par conséquent, l'heure et la longitude.

Nous avons exposé ces généralités pour attirer l'attention des observateurs sur les problèmes que peut susciter la détermination de l'heure et provoquer ainsi les discussions scientifiques. Les comparaisons fréquentes de nos pendules modernes avec l'horloge céleste nous feront peut-être douter un jour de la régularité de la rotation de la terre. ¶ Serait-il téméraire de supposer que des variations de la pesanteur **—en

* *Ast. Jour.*, No. 774 (1920).

† W. B. Varnum, "On Differential Horizontal Refraction," *Ast. Jour.*, No. 787 (1921).

‡ E. Pasquier, "Sur les variations de la latitude et les déviations de la verticale," *Ann. Soc. scientifique de Bruxelles* (1911–1912).

§ M. Alliaume, "La rigidité de la terre d'après des expériences récentes," *Rev. des Questions scientifiques*, t. 39 (1921).

|| H. Bouasse, *Géographie mathématique*, p. 334.

¶ H. Glauert, *M.N.R.A.S.*, 85, 489 (1915).

** Professor Sampson, "Studies in Clocks and Time-keeping," *Proc. R.S. Edin.*, 38, 75 (1918).

direction et en intensité—et les petites secousses sismiques modifient les marches des horloges?*

Pour essayer de répondre à toutes ces questions l'astronome devra s'efforcer de parfaire ses méthodes de détermination et de conservation de l'heure.

Détermination astronomique de l'Heure.—Les observations méridiennes sont, à cette époque, les plus propres à la détermination de l'heure de précision. Dans les observatoires bien outillés on est près d'atteindre le centième de seconde de temps. Il ne parait pas impossible cependant de réaliser un progrès par l'application d'une autre méthode comme celle, par exemple, des hauteurs égales. Les beaux résultats obtenus par l'astrolabe à prisme de MM. Claude et Driencourt† permettent des espoirs.‡

C'est la correction du chronographe que l'on détermine par les observations astronomiques. Le chronographe est synchronisé par une pendule à température et à pression constantes et l'on s'assure que sa correction ne change pas par rapport à cette pendule (décalage) au cours d'une soirée en comparant le chronographe avant et après chaque séance d'observation. Ces deux comparaisons, qui se font automatiquement au centième de seconde près, permettent de vérifier si le retard des marteaux ne s'est pas modifié par suite d'une variation du voltage de la batterie alimentant le chronographe imprimant.

Aux instruments méridiens munis d'un micromètre enregistreur et d'un bon chronographe, on a atteint, dans la détermination des heures de passage des astres, une précision qu'il semble bien difficile de dépasser par des observations visuelles.§

Pour les observations de grande précision, certains observatoires ont abandonné le chronographe imprimant: la complexité de son mécanisme et l'imperfection de sa synchronisation lui font préférer le chronographe à plume, moins commode sans doute, mais plus précis.

Mais c'est surtout sur les corrections instrumentales de la lunette méridienne que l'observateur doit principalement porter son attention.

Collimation.—Rappelons que la collimation est l'angle que fait l'axe optique de la lunette avec un plan perpendiculaire à l'axe de rotation; c'est, en d'autres mots, la distance angulaire du fil milieu idéal au fil sans collimation. Cette correction instrumentale est inhérente à l'instrument, indépendante de son orientation. La position du fil milieu idéal se détermine en faisant la moyenne des lectures micrométriques qui correspondent aux fermetures du circuit par le micromètre enregistreur. On s'assurera que les contacts supplémentaires que l'on utilise pour l'observation des circompolaires ne

* H. Kienle, *ouv. cité.* K. Sotome, "A System of Time-keeping specially adapted to a Locality where Earthquakes are frequent," *A.N.*, No. 3981. "Studies on Astro. Time-keepers," *Ann. obser. Tokyo*, t. v. fasc. 4 (1921).

† *Conférence inter. de l'heure*, Paris, 1912, p. 55. Mme. E. Chandon, "Observations faites à l'astrolabe à prisme de MM. Claude et Driencourt," *B.A.*, nov.-déc. 1917.

‡ Claude et Driencourt, "Astrolabe à prisme," *C.R.*, No. 18, 1920. R. Baillaud, "Sur un astrolabe photographique," *C.R.*, t. 158, p. 1249.

§ H. Philippot, "L'application de la photographie aux observations méridiennes," *Bul. Soc. Belge d'Ast.*, No. 4 (1906).

modifient pas la position de ce fil milieu idéal et l'on tiendra compte de la différence, le cas échéant.

Pour la détermination du "fil sans collimation" la méthode des collimateurs horizontaux nous donne, à Uccle, une très grande précision : la collimation y est obtenue à quelques millièmes de seconde prés.

Il est prudent de vérifier de temps en temps les collimations ainsi obtenues par un retournement de la lunette sur une mire ou sur une circompolaire, mais nous pensons qu'il faut se garder de trop tourmenter la lunette par de fréquents retournements. Lorsque l'objectif et le micromètre sont bien serrés, la collimation garde longtemps la même valeur.

L'inclinaison.—Varie parfois au cours d'une soirée sous l'influence de la température, de l'humidité de l'air et des mouvements du sol. Il est donc bon de la mesurer plusieurs fois au cours d'une même soirée et de tenir compte, éventuellement, de ses variations. A l'observatoire d'Uccle, nous avons abandonné l'usage du niveau, les mesures de l'inclinaison par le bain de mercure nadiral nous inspirant beaucoup plus de confiance. Pour supprimer l'influence de la tension superficielle et être sûr ainsi de l'horizontalité de la surface réfléchissante, nous utilisons une couche de mercure épaisse de trois millimètres au moins. Les observatoires dont le sol subit de continuelles trépidations causées par le passage des voitures dans les rues voisines, ont grand avantage à se servir du bain de mercure de M. Hamy qui amortit les oscillations.

La *déviation azimutale* (azimut) est la correction instrumentale dont la mesure offre le plus de difficultés. Aucun procédé physique ne permet de l'obtenir avec précision et il faut donc toujours avoir recours à des observations d'étoiles. On sait que chaque observation méridienne fournit une équation de la forme

$$a \,.\, \sin(\phi - \delta) \sec \delta + c_p = \alpha - h,$$

où a est l'azimut à déterminer, α et δ l'ascension droite et la déclinaison apparentes (empruntées aux éphémérides) de l'étoile observée, ϕ la latitude du lieu d'observation, c_p la correction inconnue de la pendule, h l'heure du passage de l'étoile au fil milieu, corrigée de l'influence de la collimation et de l'inclinaison. Il est facile de voir que l'on obtiendra la déviation azimutale a par l'observation de deux étoiles de déclinaisons différentes : il sera avantageux de choisir une circompolaire et une équatoriale. La résolution des équations montre que, toutes choses égales d'ailleurs, il faut préférer un passage inférieur d'une circompolaire à un passage supérieur. On obtient aussi une bonne valeur de l'azimut en combinant entre elles les équations de deux circompolaires l'une passant au-dessus, l'autre en-dessous du pôle. L'avantage de cette combinaison d'équations n'est cependant pas tout à fait aussi grand qu'on pourrait le croire à cause de l'incertitude de la quantité observée h et des ascensions droites des deux circompolaires.

Mais il est possible * de déterminer un azimut *absolu*, indépendant des éphémérides, en combinant les équations qui correspondent à deux

* P. Stroobant, "La mesure de l'ascension droite des astres et l'usage des mires méridiennes," *Annuaire obs. roy. de Belg.*, 1904.

passages consécutifs d'une même circompolaire (passage supérieur et passage inférieur). On n'empruntera alors aux éphémérides que la *variation* de l'ascension droite dans l'intervalle de 12 heures. Cette méthode donne d'excellents résultats quand on est sûr de la marche de la pendule et que l'on peut, en pointant de bonnes mires, mesurer la variation éventuelle de l'azimut dans l'intervalle de 12 heures. Ces doubles passages d'une circompolaire s'observent particulièrement bien en hiver quand la longueur des nuits permet d'observer les deux passages dans des conditions semblables.

On peut aussi déterminer l'azimut par l'observation d'une *polarissime*, étoile dont la distance polaire n'est que de quelques minutes. A l'observatoire d'Uccle* nous observons régulièrement la polarissime B.D. + 89° 37 dont M. Courvoisier publie chaque année les éphémérides.† Ces éphémérides donnent pour chaque jour, les co-ordonnées curvilignes rectangulaires de cette étoile rapportée à deux axes invariablement liés à la sphère céleste, axes ayant le pôle pour origine et dont l'un est le cercle horaire du point vernal vrai. Connaissant l'heure sidérale approximative, on peut, à tout instant, calculer la position de ces axes, mobiles avec la sphère céleste, par rapport à des axes fixes rectangulaires ayant même origine et dont l'un est le méridien du lieu d'observation. Il sera donc facile de calculer les co-ordonnées rectangulaires instantanées de la polarissime par rapport à ces axes. La comparaison de ces co-ordonnées avec celles obtenues au micromètre (corrigées au préalable de l'influence de la collimation et de l'inclinaison) permet de déduire la déviation azimutale de la lunette. Si les éphémérides sont exactes on obtient ainsi des valeurs *instantanées* (mais non absolues) de l'azimut, qui conviennent pour le moment de l'observation.

Ayant déterminé l'azimut par une de ces méthodes, on s'assurera de la manière suivante de l'exactitude de la valeur adoptée. Désignant par Δa la correction éventuelle à appliquer à la valeur adoptée, chaque étoile nous fournira une équatione de la forme

$$k\Delta a + c_p = \mathrm{H},$$

où k et H sont connus. Faisons un diagramme en portant, en abscisses les valeurs de k et en ordonnées les valeurs de H ; nous aurons autant de points que d'étoiles observées et ces points seront distribués d'autant mieux suivant une droite que les observations sont plus précises. Si l'azimut adopté est exact cette droite sera parallèle à l'axe des abscisses ($\Delta a = 0$) (nous admettons que c_p est constant au cours de l'observation) ; si Δa est sensible, la droite sera inclinée et de cette inclinaison on pourra conclure Δa. Pour que Δa soit bien déterminé il est nécessaire d'observer des étoiles convenablement choisies sur tout le méridien : étoiles australes, zénithales et passages inférieurs.

La valeur de l'azimut ainsi améliorée sera substituée dans les équations des étoiles *équatoriales* qui donneront une correction de pendule trés voisine de l'exactitude.

Collimation, inclinaison et déviation azimutale ne sont déterminées

* H. Philippot et E. Delporte, "Observations de Nova Aquilæ," *Ann. obs. roy. de Belg.*, tome XIV., p. 26.

† Courvoisier, "Ephemeride der Polarissima," *A.N.*, Nos. 4650, 4702, 4777, . . .

que pour des positions particulières de la lunette : horizontale pour la collimation, verticale pour l'inclinaison, parallèle à l'axe du monde pour l'azimut.*

Il faudrait, par conséquent, faire intervenir les corrections éventuelles dues aux irrégularités des tourillons.† Il serait également désirable de s'assurer que la lunette ne subit pas de flexion latérale variable avec l'inclinaison de la lunette.

Remarquons en terminant que quand on a particulièrement en vue l'étude des marches des pendules il est avantageux, pour s'affranchir des erreurs d'éphémérides, de prendre, autant que possible, les mêmes étoiles à chaque séance d'observation.

Détermination de l'heure la plus probable à l'aide de plusieurs pendules.—Un autre problème se pose maintenant : Comment déterminer l'heure (la correction de la pendule principale) les jours où les observations astronomiques sont impossibles? Si l'on pouvait répondre de la parfaite régularité de la marche d'une pendule, il suffirait de faire une *interpolation* ou *extrapolation* linéaire en partant des états déterminés par des observations d'étoiles. On n'a, malheureusement, jamais cette assurance : une horloge, si bonne soit-elle, a toujours de petits écarts de marche et il arrive même que sa marche change capricieusement de sens. La correction obtenue par une seule pendule peut donc être erronée et il est prudent de *tenir compte des heures marquées par toutes les bonnes horloges dont on dispose.* À l'observatoire d'Uccle nous déterminons la correction la plus probable de la pendule principale par le procédé graphique suivant : Pour chacune des quatre pendules à pression et à température constantes nous faisons un diagramme en portant le temps en abscisse (1 cm. pour 1 jour) et les corrections de pendules en ordonnée (1 mm. pour $0^s.01$). Quand l'état du ciel le permet, l'observation astronomique nous donne la correction du chronographe et nous en déduisons, par comparaison, les corrections correspondantes des quatre pendules fondamentales. Chacune de ces corrections est réprésentée par un point dans son diagramme (point observé). Si les observations astronomiques étaient parfaites et si ces horloges avaient des marches uniformes, les points représentatifs d'une même pendule se trouveraient en ligne droite. Il n'en est jamais ainsi et l'on obtient une ligne brisée en joignant les points succesifs.

Admettons provisoirement que les observations astronomiques soient parfaites ; alors les points du diagramme représentent exactement les corrections pour les jours d'observation. La ligne qui passe par ces points nous donne, pour chaque instant, les états probables *interpolés* de chaque pendule et comme tous les garde-temps sont comparés quotidiennement à la pendule principale, nous pouvons conclure, de chacun de ces états, une correction probable de la pendule principale. C'est en adoptant l'heure ainsi interpolée que l'on établit les bulletins définitifs de marche des pendules. Pour la détermination de l'heure extrapolée (heure prévue) on prolonge chaque courbe (en s'inspirant

* M. Hamy, "Sur l'étude des perturbations de l'axe optique d'une lunette méridienne en direction," *C.R.*, 3 et 10 mars 1919.

† H. Philippot, "La forme des tourillons du cercle méridien de Repsold," *Annales obs. royal de Belgique*, 3e série, tome i (1920).

de leur allure pendant la période précédente) au-delà des points qui correspondent à la dernière détermination astronomique.

Toutes les pendules collaborent ainsi à la détermination de l'état de l'une d'elles considérée comme pendule principale. L'expérience a montré la valeur de chaque garde-temps et d'après cela nous attribuons des poids différents aux corrections qu'ils nous fournissent.

On peut aussi déterminer l'heure *extrapolée* la plus probable par le calcul : soient A_1, B_1, C_1, D_1 les heures marquées par les quatre pendules au moment de la comparaison et A_2, B_2, C_2, D_2 les heures marquées 24 heures plus tard. Formons les différences

$$\begin{array}{lll} A_1 - B_1 = \alpha_1 & A_1 - C_1 = \alpha_2 & A_1 - D_1 = \alpha_3 \\ A_2 - B_2 = \beta_1 & A_2 - C_2 = \beta_2 & A_2 - D_2 = \beta_3 \end{array}$$

puis les suivantes :

$$\begin{aligned} (A_1 - A_2) - (B_1 - B_2) &= \alpha_1 - \beta_1 = \alpha \\ (A_1 - A_2) - (C_1 - C_2) &= \alpha_2 - \beta_2 = \beta \\ (A_1 - A_2) - (D_1 - D_2) &= \alpha_3 - \beta_3 = \gamma \end{aligned}$$

et posons :

$$(A_2 - A_1) = a, \quad (B_2 - B_1) = b, \quad \text{etc.} ;$$

il vient :

$$\left.\begin{aligned} a &= b - \alpha \\ a &= c - \beta \\ a &= d - \gamma \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Or a, b, c, d représentent les marches diurnes des pendules ; si nous remplaçons donc dans ces dernières égalités b, c, d par leurs valeurs, nous déduisons pour a trois valeurs correspondantes que nous désignerons par a_b, a_c, a_d ; en y adjoignant la valeur $a_a = a$, nous aurons

$$\begin{aligned} a_a &= a \\ a_b &= b - \alpha \\ a_c &= c - \beta \\ a_d &= d - \gamma. \end{aligned}$$

Nous obtenons ainsi, pour la pendule principale A, quatre valeurs distinctes de sa marche diurne dont il s'agit de prendre la moyenne ; en attribuant les poids p_1, p_2, p_3, p_4 à ces valeurs, on prendra pour moyenne

$$a_m = \frac{p_1 a_a + p_2 a_b + p_3 a_c + p_4 a_d}{p_1 + p_2 + p_3 + p_4}.$$

Les heures A, B, C, D marquées par les quatre pendules *au même instant,* nous sont fournies par la comparaison directe de ces pendules avec le chronographe imprimant de Gautier (comparaison automatique) ; le temps nécessaire à cette opération n'étant que de quelques minutes on néglige la différence de marche pendant ce court intervalle.

Désignons par G l'heure marquée par le chronographe au moment de la comparaison et formons les différences

$$A - G = h_1$$
$$B - G = h_2$$
$$C - G = h_3$$
$$D - G = h_4$$

d'où l'on tire

$$(A - G) = (B - G) = A - B = h_1 - h_2 = \text{etc.}$$

Si les quatre pendules, comme c'est le cas à Uccle, ne sont pas toutes réglées sur le même temps, il faut au préalable transformer les heures qu'elles marquent de façon à les ramener toutes au temps moyen ou au temps sidéral. En remplaçant dans les équations (1), a par la valeur a_m trouvée, on obtiendra de nouvelles valeurs de b, c, et d que l'on utilise pour les 24 heures suivantes.

Les postes radiotélégraphiques français (Paris, Lyon, Bordeaux) émettent journellement des signaux horaires rythmés qui peuvent être utiles dans la détermination de l'heure la plus probable. Nous n'entrerons pas dans le détail de leur réception ; retenons seulement qu'ils peuvent fournir une correction de la pendule principale indépendante de celles trouvées précédemment.

Si C désigne la valeur la plus probable de la correction de la pendule principale, C_0 la correction qui résulte d'une séance d'observations méridiennes, C_1, C_2, C_3, et C_4 les valeurs déduites des quatre pendules fondamentales, C_5 celle qui provient de la réception des signaux de T.S.F. (ces diverses corrections étant ramenées au même instant) et si nous appelons p_0, p_1, p_2, p_3, p_4, p_5 leurs poids respectifs, nous adopterons pour C :

$$C = \frac{p_0 C_0 + p_1 C_1 + \ldots + p_5 C_5}{p_1 + p_2 + \ldots + p_5}.$$

Les poids p_1, p_2, p_3, p_4 dépendent du fonctionnement des pendules correspondantes ; ils peuvent être calculés facilement. Néanmoins ils sont sujets à variation : ainsi quand on vient de modifier une pression sous une cloche, le poids de la pendule correspondante sera nul.

En ce qui concerne p_5, il est évidemment variable mais dans des conditions inconnues de l'observatoire récepteur ; il ne pourra donc avoir de grande valeur que si les autres p sont faibles. Pour tout observatoire bien outillé sous le rapport des pendules et de l'instrument méridien, p_5 est négligeable vis-à-vis des autres. À Uccle nous prenons toujours $p_5 = 0$; nous ne recevons les signaux rythmés de T.S.F. qu'à seule fin de déterminer leur correction par nos propres moyens.

Quand on a pu déterminer l'heure par des observations astronomiques, p_0 a évidemment une valeur prépondérante. Toutefois avant de lui donner une valeur définitive, il convient de voir si la correction C_0 déduite de l'observation n'a pas une valeur anormale, de tenir compte également du nombre d'étoiles observées, de la qualité des images, de la valeur de l'azimut, etc.

Nous avons supposé que les observations astronomiques étaient parfaites, qu'elles nous avaient fourni l'heure *exacte*. Il est cependant difficile d'admettre que des horloges indépendantes * à pression et à température constantes subissent (comme le montrent parfois les graphiques ou les calculs) des perturbations sensibles de même grandeur et de même sens: on admet généralement que ces écarts sont plus apparents que réels † et qu'ils sont dus à des erreurs d'observation. Nous n'attribuons donc pas la même valeur à toutes nos déterminations astronomiques de l'heure et, soit sur les diagrammes soit dan les calculs, nous attachons moins d'importance aux observations qui donnent des corrections manifestement anormales. Dans ce cas, on s'inspire plutôt de la marche des pendules pendant la période précédente.

Uccle, Observatoire royal de Belgique:
1921 *novembre.*

Observatoire Royal de Belgique: Note sur la réception des signaux horaires scientifiques de Paris.

(*Communicated by Professor R. A. Sampson.*)

La réception des signaux horaires scientifiques de Paris est généralement faite, à Uccle, par la comparaison des battements rythmés de la Tour Eiffel avec ceux de la pendule sidérale No. 7 servant aux observations de passage.‡ L'intervalle de temps qui sépare deux battements est donc exprimé en seconde sidérale.

Le chronographe imprimant de Gautier, dont on détermine la correction par les observations méridiennes, est synchronisé par la pendule No. 7. Par une dérivation mise sur le circuit de cette synchronisation, les battements de la pendule sont reproduits dans le cornet du téléphone de T.S.F. Ce procédé permet donc de comparer directement les battements de T.S.F. avec ceux de la pendule d'observation et d'éliminer ainsi les erreurs pouvant provenir de comparaisons intermédiaires. Nous avons tenu compte du retard des marteaux du chronographe pour passer de la correction du chronographe à celle qui correspond aux battements de la pendule entendus dans le cornet du téléphone de T.S.F. Ce retard est constant et est de $0^s.02$.

Nous ne nous sommes servi qu'occasionnellement d'une pendule ou d'un chronomètre de temps moyen. Dans ce cas l'intervalle de deux battements est exprimé en seconde de temps moyen (s.m.).

Les déterminations de l'heure ont été faites au cercle méridien de

* Les supports des pendules doivent être indépendants et les plans d'oscillation seront différemment orientés.

† Si ces perturbations étaient *réelles*, elles seraient révélées sans doute par leur retour périodique. Par l'étude de cette périodicité, l'observateur essayerait de discerner si ces perturbations ont pour cause une variation de la pesanteur (pendules) ou une variation du méridien (observations astronomiques).

‡ Voir *Description des installations du service de l'heure*, par H. Philippot et E. Delporte.

Repsold de $1^m \cdot 90$ de distance focale et de 16 centimètres d'ouverture, muni d'un micromètre enregistreur. La collimation a toujours été déterminée à l'aide de collimateurs, l'inclinaison par le bain de mercure et la déviation azimutale par l'observation de la polarissime B.D. $+89^\circ 37$ et, de temps en temps par double passage d'une circompolaire. Les observations pour l'heure ont toujours été faites par le même observateur, M. F. Moreau.

Les dates en italiques sont celles où l'on a obtenu des observations méridiennes. Pour les autres dates la correction de la pendule sidérale No. 7 a été déduite par la comparaison des quatre pendules fondamentales indépendantes à pression et à température constantes dont les marches sont petite et régulières (marche moyenne, $0^s \cdot 04$ au maximum ; écart moyen de la marche, $\pm 0^s \cdot 01$ environ). Ces comparaisons sont faites à l'aide du chronographe de Gautier sur lequel les pendules comparées inscrivent automatiquement l'heure qu'elles marquent.

La conversion du temps sidéral en temps moyen a toujours été faite utilisant le temps sidéral à midi moyen de Greenwich indiqué par le *Nautical Almanac*. (Ce temps sidéral est $0^s \cdot 06$ plus petit que celui que donne la *Connaissance des Temps.*)

Les résultats qui suivent correspondent à une *heure extrapolée de Paris* et à une *heure interpolée d'Uccle.* Les heures renseignées pour l'Observatoire d'Uccle sont donc définitives ; celles de Paris doivent *encore être corrigées.*

Date 1920.	Heure Greenwich.	U.–P. 1er.	U.–P. 300e.
Avril		s	s
15	23	+ ·08	+ ·11
16	10½	+ ·01	+ ·03
16	23	·00	+ ·02
16	23½	+ ·01	+ ·02
17	10½	− ·05	− ·02
17	23	− ·01	− ·01
19	10½	− ·04	− ·02
19	23	− ·06	− ·06
19	23½	− ·03	− ·03
20	10½	− ·11	− ·05
21	,,	− ·20	− ·14
21	23	− ·13	− ·15
21	23½	− ·12	− ·14
22	10½	− ·19	− ·10
22	23	− ·16	− ·15
23	10½	− ·13	− ·11
24	23	− ·11	− ·04
24	23½	− ·08	− ·04
26	10½	− ·10	− ·08
Avril		s	s
26	23	− ·04	− ·04
26	23½	− ·09	− ·11
28	10½	− ·03	·00
28	23	− ·05	− ·08
28	23½	− ·11	− ·10
29	10½	− ·09	− ·09
29	23	− ·18	− ·20
29	23½	− ·15	− ·12
30	10½	− ·17	− ·13
Mai			
3	10½	− ·02	− ·03
5	,,	− ·04	− ·01
5	23	+ ·02	− ·02
5	23½	− ·05	·00
6	10½	+ ·06	− ·01
6	23	− ·06	+ ·01
7	10½	+ ·09	+ ·09
7	23	+ ·08	+ ·08
7	23½	− ·01	+ ·03
Mai		s	s
8	10½	+ ·04	+ ·06
10	,,	+ ·12	+ ·14
10	23	+ ·10	+ ·09
11	10½	+ ·15	+ ·15
11	23	+ ·10	+ ·08
12	10½	+ ·03	+ ·04
14	,,	+ ·09	+ ·09
14	23	+ ·11	+ ·09
15	10½	+ ·15	+ ·08
17	,,	+ ·07	+ ·08
18	,,	+ ·04	+ ·06
19	,,	+ ·02	+ ·05
20	23	− ·03	·00
21	10½	− ·03	− ·02
22	23	− ·02	·00
25	10½	− ·06	− ·05
27	,,	− ·05	− ·09
27	23	− ·05	− ·02
27	23½	− ·07	− ·06
Mai		s	s
28	10½	− ·16	− ·11
28	23	− ·16	− ·12
28	23½	− ·17	− ·13
29	23	− ·12	− ·10
31	10½	− ·14	− ·12
31	23	− ·10	− ·13
Juin			
1	10½	− ·09	− ·12
2	,,	− ·08	− ·03
2	23	− ·01	+ ·03
2	23½	− ·07	− ·03
3	23	− ·02	− ·05
4	10½	− ·06	− ·03
7	,,	− ·09	− ·04
7	23½	− ·06	− ·01
9	10½	− ·03	− ·10
9	23	+ ·02	·00
9	23½	− ·07	− ·04
10	10½	+ ·01	− ·04

Date 1920.	Heure Green-wich.	U.–P. 1er.	U.–P. 300e.
		s	s
Juin			
11	10½	– ·06	– ·01
12	,,	– ·03	– ·02
14	,,	– ·07	– ·02
14	23	– ·08	– ·02
14	23½	– ·04	– ·06
15	10½	– ·04	– ·08
16	,,	– ·07	– ·14
16	,,	– ·06	– ·12
16	23½	– ·11	– ·05
18	10½	– ·09	– ·12
19	,,	·00	– ·06
21	,,	– ·01	– ·06
21	23	– ·03	– ·01
21	23½	– ·07	– ·02
23	10½	– ·04	– ·08
23	23½	·00	– ·02
24	10½	+ ·09	+ ·06
24	23	– ·06	– ·08
24	23½	+ ·05	+ ·05
25	10½	+ ·04	·00
25	,,	+ ·01	– ·03
26	,,	+ ·07	+ ·11
28	,,	+ ·07	+ ·10
29	,,	– ·01	+ ·02
29	,,	+ ·06	+ ·04
30	,,	– ·05	– ·02
30	23	– ·03	– ·08
30	23½	– ·07	– ·05
Juil.			
1	10½	– ·03	+ ·02
1	23	+ ·04	+ ·03
1	23½	+ ·05	+ ·10
2	10½	– ·01	– ·05
2	23	– ·03	+ ·02
2	23½	– ·07	– ·02
3	10½	– ·05	– ·04
6	,,	– ·07	– ·04
7	,,	– ·11	– ·14
7	23½	– ·14	– ·17

Date 1920.	Heure Green-wich.	U.–P. 1er.	U.–P. 300e.
		s	s
Juil.			
8	10½	– ·11	– ·14
9	,,	– ·20	– ·16
10	,,	+ ·01	+ ·03
10	23	– ·04	·00
10	23½	– ·04	– ·02
13	10½	·00	– ·04
14	,,	– ·07	– ·04
15	,,	– ·08	– ·06
15	23	+ ·03	+ ·02
16	10½	·00	– ·04
17	,,	– ·01	·00
19	,,	– ·07	– ·07
24	,,	– ·33	– ·28
26	,,	– ·34	– ·31
27	,,	– ·49	– ·55
28	,,	– ·73	– ·78
28	,,	– ·71	– ·77
28	23½	– ·17	– ·17
29	10½	– ·15	– ·13
31	,,	$-6^{m}\cdot42\cdot19$	$-6^{m}\cdot42^{s}\cdot23$
31	,,	$-6^{m}\cdot42\cdot10$	$-6^{m}\cdot42^{s}\cdot16$
Août			
2	10½	– ·13	– ·14
2	,,	– ·20	– ·20
3	,,	– ·19	– ·16
3	,,	– ·19	– ·23
5	,,	– ·25	– ·24
6	,,	– ·19	– ·19
9	,,	– ·06	– ·04
11	,,	– ·05	– ·02
14	,,	– ·08	– ·06
16	,,	– ·10	– ·08
17	,,	– ·10	– ·06
18	23	– ·05	– ·06
19	10½	– ·07	– ·09
19	23½	– ·12	– ·16
21	10½	– ·11	– ·18
24	,,	+ ·05	+ ·02
25	,,	– ·05	·00
27	,,	– ·03	·00

Date 1920.	Heure Green-wich.	U.–P. 1er.	U.–P. 300e.
		s	s
Août			
25	23	– ·03	+ ·01
25	23½	+ ·03	– ·01
26	10½	– ·01	+ ·03
28	,,	·00	+ ·07
30	,,	– ·03	– ·03
31	,,	– ·08	– ·09
Sept.			
1	23	– ·13	– ·19
1	23½	– ·17	– ·17
2	10½	– ·16	– ·18
2	23	– ·22	– ·23
2	23½	– ·22	– ·27
3	10½	– ·17	– ·23
4	,,	– ·33	– ·35
6	,,	– ·25	– ·27
7	,,	– ·31	– ·35
8	,,	– ·34	– ·37
9	,,	– ·33	– ·33
9	23	– ·09	– ·07
9	23½	– ·16	– ·10
10	10½	+ ·04	+ ·06
13	,,	+ ·10	+ ·12
15	,,	+ ·17	+ ·13
16	,,	+ ·22	+ ·18
16	23½	+ ·16	+ ·10
17	10½	+ ·14	+ ·18
18	,,	+ ·15	+ ·19
19	23	+ ·19	+ ·17
19	23½	+ ·14	+ ·18
20	10½	+ ·07	+ ·07
21	,,	+ ·01	+ ·05
21	,,	+ ·05	+ ·09
22	,,	– ·03	– ·11
23	,,	+ ·01	– ·04
23	23	– ·04	– ·07
23	23½	– ·04	– ·10
24	10½	– ·08	– ·05
25	,,	– ·05	·00
27	,,	– ·03	·00

Date 1920.	Heure Green-wich.	U.–P. 1er.	U.–P. 300e.
		s	s
Sept.			
28	10½	– ·02	– ·0
29	,,	– ·05	– ·0
30	,,	– ·01	– ·0
Oct.			
1	10½	+ ·11	+ ·0
2	,,	+ ·10	+ ·0
4	,,	+ ·07	+ ·0
5	,,	+ ·11	+ ·0
5	,,	+ ·09	+ ·0
6	,,	– ·03	·0
7	,,	+ ·02	– ·0
8	,,	– ·04	– ·0
9	,,	– ·11	– ·0
11	,,	+ ·02	– ·0
12	,,	– ·04	– ·0
12	23	– ·07	– ·1
12	23½	– ·10	– ·0
13	10½	·00	+ ·0
14	,,	– ·02	– ·0
15	,,	– ·06	– ·1
16	,,	– ·06	– ·0
16	23	– ·11	– ·0
16	23½	– ·12	– ·1
18	10½	+ ·03	+ ·0
18	23	– ·09	– ·0
18	23½	– ·13	– ·1
19	10½	– ·06	– ·0
21	,,	– ·09	– ·0
22	,,	– ·15	– ·1
23	,,	– ·08	– ·1
25	,,	– ·12	– ·1
25	23	– ·04	·0
25	23½	– ·01	– ·0
26	10½	– ·08	– ·0
27	,,	– ·07	– ·0
27	23	+ ·01	+ ·0
27	23½	– ·03	– ·0
28	10½	– ·01	– ·0
29	,,	– ·05	– ·0

Date 1920.	Heure Green-wich.	U.-P. 1er.	U.-P. 300e.
ct.		s	s
29	23	·00	−·01
29	23½	−·07	−·04
30	10½	+·13	+·06
ov.			
3	10½	−·10	−·12
4	,,	−·10	−·13
5	,,	−·13	−·10
5	23	·00	−·06
5	23½	·00	−·08
6	10½	−·07	−·07
8	,,	−·16	−·18
9	,,	−·12	−·14
11	,,	−·01	−·07
12	,,	−·03	−·01
16	,,	−·01	+·01
17	,,	·00	+·03
18	,,	+·01	−·02
18	23	−·02	+·03
18	23½	−·03	−·01
19	10½	·00	+·01
22	,,	+·06	+·05
23	,,	+·10	+·10
23	23	−·01	·00
24	10½	+·01	+·04
25	,,	−·01	·00
25	23	+·04	−·03
26	10½	−·01	−·07
27	,,	−·12	−·05
29	,,	−·16	−·12
éc.			
1	10½	−·19	−·20
2	,,	−·15	−·21
3	,,	−·21	−·26
3	23	−·24	−·29
3	23½	−·23	−·25
4	10½	−·26	−·28
6	,,	−·06	−·02
7	,,	−·15	−·11
8	,,	−·09	−·07

Date 1920.	Heure Green-wich.	U.-P. 1er.	U.-P. 300e.
Déc.		s	s
9	10½	−·01	+·01
10	,,	+·06	+·03
11	,,	−·04	+·01
13	,,	+·07	+·13
14	,,	+·09	+·06
15	,,	−·04	−·02
17	,,	−·08	−·14
17	23	−·02	−·01
17	23½	−·08	−·02
18	,,	−·03	−·05
22	10½	+·06	+·04
23	,,	+·13	+·09
24	,,	+·14	+·14
27	,,	+·21	+·23
28	23	+·19	+·16
29	10½	+·19	+·23
30	,,	+·23	+·19
31	,,	+·15	+·18
1921. Janv.			
3	10½	+·15	+·13
4	,,	+·07	+·04
5	,,	+·25	+·19
6	,,	+·16	+·15
7	,,	+·20	+·19
8	,,	+·19	+·19
10	,,	+·15	+·10
11	,,	+·10	+·09
12	,,	+·05	+·10
13	,,	+·05	·00
14	,,	−·05	−·04
15	,,	+·04	+·06
17	,,	−·08	−·08
18	,,	−·13	−·11
19	,,	−·15	−·12
20	,,	−·14	−·15
22	,,	−·14	−·16
24	,,	−·19	−·18
25	,,	−·19	−·20

Date 1921.	Heure Green-wich.	U.-P. 1er.	U.-P. 300e.
Janv.		s	s
26	10½	−·17	−·15
27	,,	−·24	−·25
28	,,	−·16	−·13
29	,,	−·20	−·22
31	,,	−·24	−·25
Févr.			
1	10½	−·27	−·26
2	,,	−·25	−·25
3	,,	−·23	−·15
4	,,	−·17	−·18
5	,,	−·16	−·18
7	,,	−·12	−·14
8	,,	−·14	−·14
9	,,	−·08	−·08
10	,,	−·09	−·10
11	,,	−·07	−·08
12	,,	−·07	−·07
14	,,	−·12	−·14
15	,,	−·09	−·12
16	,,	−·14	−·14
17	,,	−·15	−·17
18	,,	−·30	−·31
19	,,	−·09	−·11
21	,,	−·14	−·14
22	,,	+·16	+·16
23	,,	−·03	−·04
24	,,	−·03	−·04
25	,,	−·01	−·03
26	,,	+·03	+·02
28	,,	+·01	−·01
Mars			
1	10½	+·06	+·05
2	,,	+·05	+·04
3	,,	+·10	+·10
4	,,	+·04	+·01
5	,,	−50·16	+9·83
7	,,	+·06	+·05
8	,,	+·05	+·06
10	,,	+·10	+·06

Date 1921.	Heure Green-wich.	U.-P. 1er.	U.-P. 300e.
Mars		s	s
11	10½	+·14	+·15
12	,,	+·13	+·12
14	,,	+·11	+·12
15	,,	+·10	+·08
17	,,	+·04	+·01
18	,,	−·02	−·01
19	,,	−·05	−·07
21	,,	−·07	−·08
23	,,	−·07	−·09
24	,,	−·10	−·08
25	,,	−·08	−·08
26	,,	−·07	−·08
29	,,	−·07	−·08
30	,,	+·01	−·01
31	,,	−·02	−·02
Avril			
1	10½	+·01	+·02
2	,,	−·06	−·06
4	,,	−·09	−·10
5	,,	−·13	−·14
6	,,	−·14	−·15
7	,,	−·13	−·16
8	,,	−·14	−·15
9	,,	−·12	−·13
11	,,	−·13	−·14
12	,,	−·17	−·19
13	,,	−·13	−·13
14	,,	−·12	−·13
15	,,	−·06	−·03
16	,,	−·08	−·09
18	,,	−·05	−·05
19	,,	−·07	−·06
20	,,	−·03	−·04
21	,,	+·06	+·02
22	,,	−·04	−·05
23	,,	+·14	+·14
25	,,	−·01	−·01
26	,,	−·01	−·02
27	,,	+·05	·00

Date 1921.	Heure Greenwich.	U.–P. 1er.	U.–P. 300e.
		s	s
Avril			
28	10½	+ ·01	+ ·02
29	,,	– ·01	– ·05
30	,,	·00	+ ·01
Mai			
2	10½	– ·13	– ·14
3	,,	– ·05	– ·06
4	,,	– ·11	– ·12
6	,,	– ·13	– ·13
7	,,	– ·15	– ·13
9	,,	– ·19	– ·20
10	,,	– ·13	– ·08
11	,,	– ·18	– ·16
12	,,	– ·09	– ·09
13	,,	– ·16	– ·08
17	,,	– ·12	– ·13
18	,,	– ·08	– ·11
19	,,	– ·09	– ·09
20	,,	– ·09	– ·10
21	,,	– ·09	– ·07
23	,,	– ·16	– ·17
24	,,	– ·17	– ·18
25	,,	– ·16	– ·16
26	,,	– ·10	– ·11
27	,,	– ·10	– ·11
28	,,	– ·10	– ·09
30	,,	– ·08	– ·10
31	,,	– ·10	– ·12
Juin			
1	10½	– ·15	– ·16
2	,,	– ·13	– ·14
3	,,	– ·17	– ·14
Juin			
6	10½	– ·14	– ·11
7	,,	– ·15	– ·16
8	,,	– ·20	– ·17
9	,,	– ·18	– ·19
10	,,	– ·18	– ·18
11	,,	– ·14	– ·15
13	,,	– ·13	– ·17
14	,,	– ·10	– ·15
15	,,	– ·10	– ·18
16	,,	– ·21	– ·19
17	,,	– ·21	– ·17
18	,,	– ·14	– ·07
20	,,	+ ·04	– ·02
21	,,	– ·01	– ·02
22	,,	– ·05	– ·01
23	,,	+ ·03	– ·01
24	,,	+ ·06	+ ·01
28	,,	– ·29	– ·25
29	,,	– ·28	– ·30
30	,,	– ·30	– ·38
Juil.			
1	10½	+ ·23	+ ·30
2	,,	– ·39	– ·46
4	,,	– ·37	– ·42
5	,,	– ·34	– ·35
6	,,	– ·29	– ·36
7	,,	– ·20	– ·24
8	,,	– ·15	– ·18
9	,,	– ·10	– ·10
12	,,	+ ·06	+ ·06
14	,,	+ ·17	+ ·07
15	,,	+ ·15	+ ·26
Juil.			
16	10½		
18	,,	+ ·04	+ ·22
19	,,	+ ·04	+ ·05
20	,,	+ ·05	+ ·12
23	,,	+ ·23	+ ·12
25	,,	+ ·14	+ ·14
26	,,	+ ·11	+ ·15
27	,,	+ ·14	+ ·13
28	,,	+ ·02	+ ·04
29	,,	+ ·05	+ ·06
30	,,	– ·04	– ·03
Août			
1	10½		
2	,,	+ ·02	+ ·01
3	,,	+ ·03	– ·03
5	,,	+ ·11	+ ·13
6	,,	– ·07	– ·05
8	,,	– ·01	– ·02
9	,,	– ·09	– ·04
10	,,	– ·03	– ·04
11	,,	– ·11	– ·11
12	,,	– ·16	– ·14
13	,,	– ·25	– ·20
16	,,	– ·28	– ·26
17	,,	– ·26	– ·25
18	,,	– ·25	– ·27
19	,,	– ·21	– ·27
20	,,	– ·26	– ·23
22	,,	– ·29	– ·23
23	,,	– ·28	– ·19
24	,,	– ·25	– ·31
25	,,	– ·29	– ·39
Août			
26	10½	– ·31	– ·3
27	,,	– ·29	– ·2
29	,,	– ·28	– ·2
30	,,	– ·27	– ·3
31	,,	– ·30	– ·3
Sept.			
1	10½	– ·34	– ·3
2	,,	– ·20	– ·1
2	,,	– ·18	– ·2
3	,,	– ·27	– ·2
5	,,	– ·34	– ·3
6	,,	– ·21	– ·1
6	,,	– ·27	– ·2
7	,,	– ·28	– ·2
8	,,	– ·22	– ·1
9	,,	– ·21	– ·1
10	,,	– ·19	– ·1
12	,,	– ·15	– ·1
13	,,	– ·16	– ·1
14	,,	– ·24	– ·2
15	,,	– ·19	– ·1
16	,,	– ·17	– ·1
17	,,	– ·19	– ·1
20	,,	– ·19	– ·1
21	,,	– ·18	– ·1
22	,,	– ·18	– ·1
23	,,	– ·20	– ·1
24	,,	– ·17	– ·1
26	,,	– ·18	– ·1
28	,,	– ·19	– ·2
29	,,	– ·22	– ·2
30	,,	– ·16	– ·1

On the Determination of Time at Different Observatories.

By Professor R. A. Sampson. (Plates 4, 5.)

Publication of times of receipt of wireless signals at different observatories permits the comparison of determinations of time at the observatories themselves, the signal itself being eliminated. And when several observatories are linked together in this way, whether by means of the same signal or of different signals, it becomes possible by a simple step to compare the time determination of each observatory with the mean of all, and so to assign to each with some considerable certainty the amount by which its own determination on any occasion was astray, at any rate in so far as fluctuations from a datum line are concerned. When this correction is accepted, it may be carried back in criticism of the Transit Circle observations, from which the time determination in question was drawn, for the purpose of searching for the origin of the discrepancies to which such observations are liable.

In 1920 November (*M.N.*, 81, 1) I published in summary the results of comparison *Paris – Edinburgh* from 1914 March to 1920 October, taken by the rhythmic signals. In the present communication I carry these forward and give more detail regarding them. The Astronomer Royal has now given us the Greenwich times of receipt of Paris signals from 1912 August and the Nauen (Berlin) and Annapolis (Washington) signals also from 1920 January to 1921 September. He has kindly supplied me with a copy of his MS., which permits the Greenwich numbers to be incorporated in the following comparisons. In taking these, various methods were employed, but evidence is given that the material is effectively homogeneous, and I accept it as such for the purpose of this paper. Further, with the courteous consent of the Director of Uccle Observatory, MM. Philippot and Moreau, who are in charge of the time service, have placed at my disposal in full detail the times of receipt at Uccle of Paris rhythmic signals from 1920 April onwards. There is thus material for comparison, which we may denote as P – E, G – P, G – B, G – W, and U – P, and by an arithmetical step which involves no obliteration of individual features we obtain B – M, E – M, G – M, P – M, U – M, W – M where $M = \frac{1}{6}(W + E + G + B + P + U)$ and B – M, . . . denotes the amount by which the time of Berlin, . . . is fast on the mean. Putting these numbers side by side, we have a chart in which the inequalities peculiar to each observatory in turn are present as to five-sixths of their full amount, qualified only by one-sixth of the mean of the other five. This is an effective isolation of each observatory, by which its own errors are assigned to it. No such step was practicable before wireless signals supplied the necessary links. For the period anterior to 1920 I have only G, P, E, compared with their own mean, so that the numbers are less discriminative; and in the case of a few short gaps in the 1920–21 series one or other of the observatories has had to be omitted and adjustment made accordingly. The method by which the numbers P – E are derived is explained in my paper of 1920 November along with other details which need not be repeated here. In the following

pages "Plotted" Clock Errors are employed. I have rearranged the material and followed the Astronomer Royal in taking means for weeks for the later period and means for months for the earlier one.

Monthly Means.

$M = \frac{1}{3}\{E+G+P\}$ (1914–1919).

$M = \frac{1}{5}(B+E+G+P+W)$ (1920–1921).

N.B.—G − M positive indicates Greenwich time fast on the mean. (The unit is 0·01 sec.)

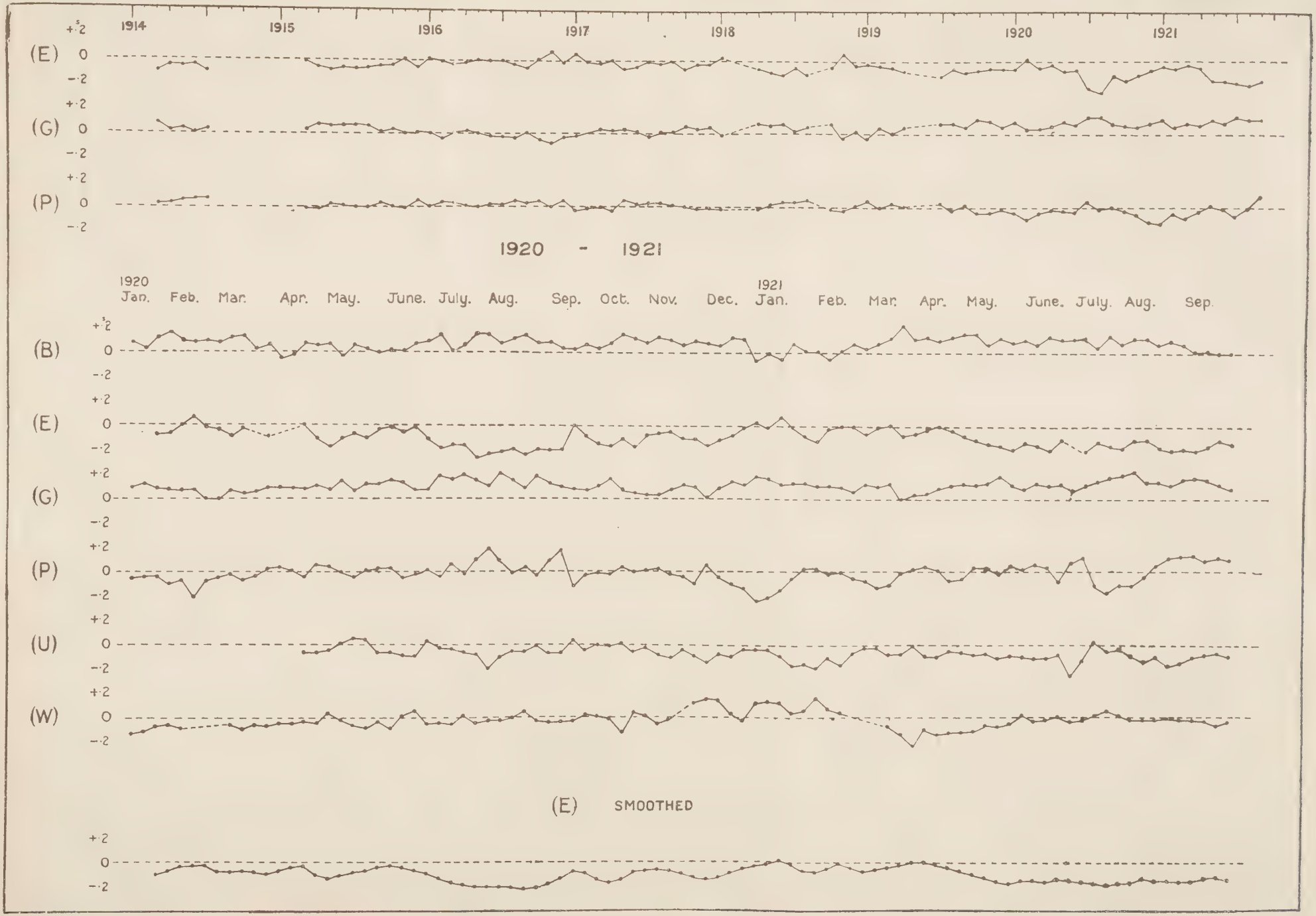
	Jan.	Feb.	Mar.	Apr.	May.	Jun.	Jul.	Aug.	Sept.	Oct.	Nov.	Dec.
1914												
E − M	...	...	−10	−6	− 7	− 6	− 9	...	...	...	...	...
G − M	...	...	8	2	3	0	3	...	...	...	...	...
P − M	...	...	2	3	5	6	6	...	...	...	...	...
1915												
E − M	...	...	− 2	−6	− 8	− 6	− 7	− 6	− 4	− 4	1	− 6
G − M	...	...	3	7	6	6	7	6	1	3	0	1
P − M	...	...	− 2	−2	2	1	0	0	3	0	− 1	5
1916												
E − M	1	− 1	− 3	−2	0	0	0	− 3	− 5	1	7	− 2
G − M	0	−4	1	2	1	− 2	− 2	− 3	1	− 5	− 8	− 3
P − M	0	4	3	1	0	2	2	5	3	5	0	5
1917												
E − M	5	− 1	− 2	0	− 7	− 5	− 1	− 3	− 1	− 7	− 3	− 3
G − M	− 2	1	3	2	3	2	− 2	1	1	6	4	5
P − M	− 3	− 1	0	−3	5	2	3	3	1	0	− 2	− 1
1918												
E − M	2	...	...	−7	−10	−12	− 6	−11	...	− 5	5	− 4
G − M	− 1	...	...	8	7	8	2	6	...	8	− 3	2
P − M	− 2	...	...	−2	2	4	4	6	...	− 2	− 3	1
1919												
E − M	− 2	−4	− 5	−8	...	...	−12	− 6	− 9	− 7	− 6	− 6
G − M	− 3	5	1	6	...	...	9	9	6	12	11	6
P − M	5	0	3	1	...	...	3	− 2	2	− 4	− 4	− 1
1920												
E − M	− 6	1	− 5	−3	− 8	− 7	−22	−25	−12	−15	−11	− 7
G − M	10	5	5	7	11	9	15	15	9	8	7	10
P − M	− 4	−9	− 4	−1	− 2	− 3	5	0	1	− 2	− 5	−11
1921												
E − M	− 4	−5	− 3	−5	−16	−16	−17	−19	−16	...	...	...
G − M	12	7	10	8	12	10	15	13	13	...	...	...
P − M	−12	−4	− 7	−2	2	0	− 6	0	10	...	...	..

Weekly Means for 1920, 1921.

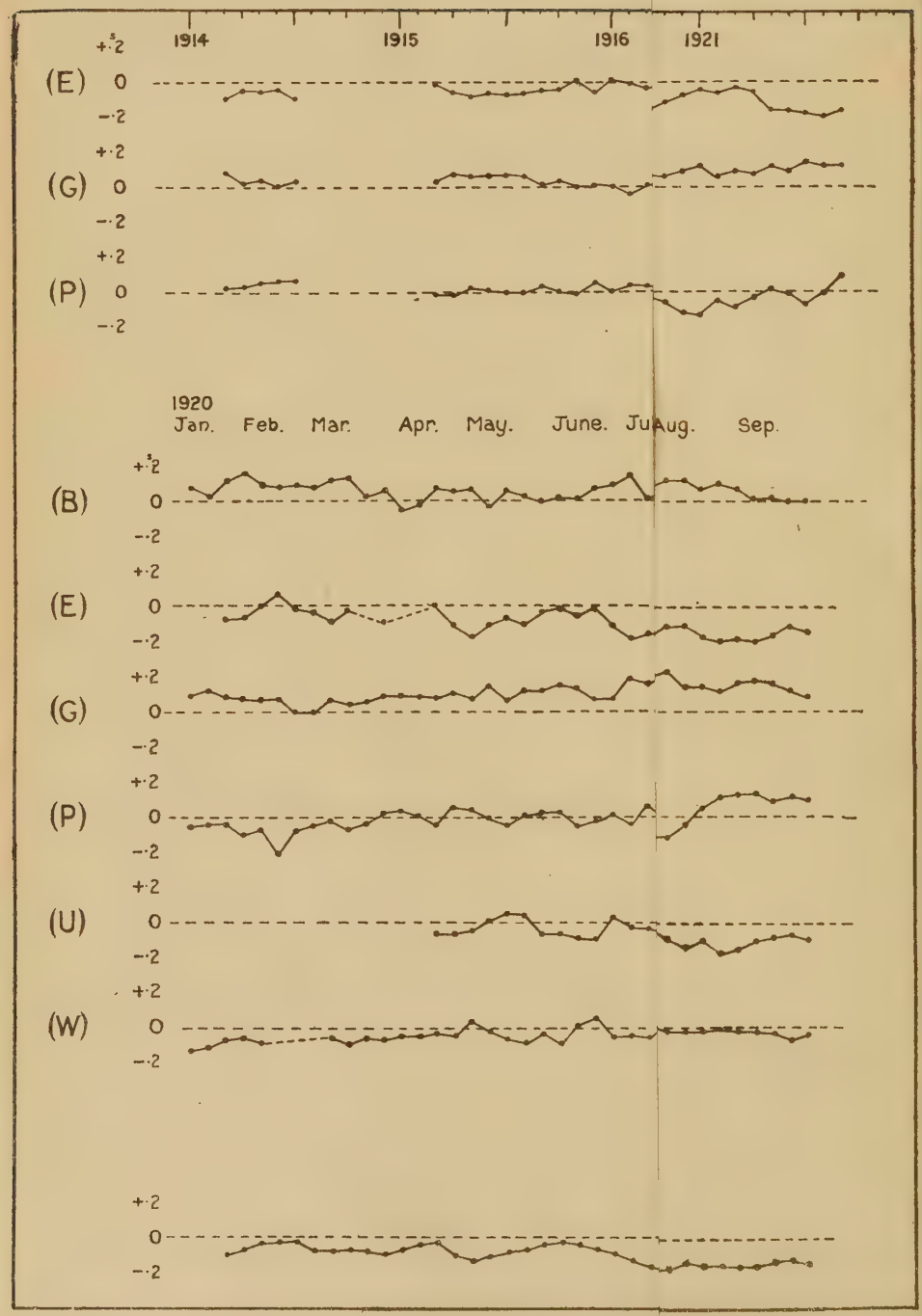
In forming these means the Greenwich numbers G – P, G – B, G – W from the Paris, Nauen, and Annapolis signals respectively are employed, in conjunction with P – E and U – P, which are derived from the receipt of the 10.30 a.m. Paris rhythmic series at Edinburgh and Uccle respectively. M denotes the mean $\frac{1}{6}$(B + E + G + P + U + W), but with adjustment when a gap occurs in the record of one or another observatory. (B) denotes B – M. (The unit is ·01 sec.)

Week commencing	(B).	(E).	(G).	(P).	(U).	(W).
1920 Jan. 5	8	...	9	– 5	...	– 13
12	3	...	12	– 4	...	– 11
19	12	– 7	8	– 4	...	– 7
26	17	– 6	7	– 10	...	– 6
Feb. 2	9	0	7	– 7	...	– 9
9	8	7	7	– 21	...	...
16	9	– 1	0	– 8	...	...
23	8	– 3	0	– 5	...	...
Mar. 1	12	– 8	7	– 3	...	– 6
8	13	– 2	5	– 7	...	– 9
15	3	...	6	– 4	...	– 6
22	6	– 9	9	2	...	– 7
29	– 6	...	9	3	...	– 5
Apr. 5	– 2	...	8	1	...	– 5
12	8	0	8	– 5	– 6	– 4
19	6	– 10	10	5	– 6	– 5
26	7	– 17	7	4	– 5	3
May 3	– 3	– 10	15	– 1	1	– 3
10	6	– 6	6	– 5	5	– 7
17	3	– 10	12	1	4	– 9
24	0	– 3	12	2	– 7	– 4
31	2	– 1	15	2	– 7	– 9
June 7	2	– 5	14	– 5	– 9	1
14	6	– 2	7	– 3	– 10	5
21	9	– 11	7	1	3	– 6
28	15	– 18	19	– 4	– 4	– 5
July 5	1	– 15	16	6	– 4	– 6
12	6	– 15	20	– 3	– 6	1
19	16	– 26	15	10	– 9	– 5
26	15	– 23	11	19	– 20	– 3
Aug. 2	8	– 21	21	9	– 11	– 3
9	12	– 19	15	– 1	– 6	– 1
16	15	– 24	9	4	– 6	4
23	8	– 21	19	– 2	– 2	– 3

Week commencing	(B).	(E).	(G).	(P).	(U).	(W).
1920 Aug. 30	9	−20	14	9	− 8	− 4
Sept. 6	4	−19	10	18	− 7	− 4
13	4	0	8	−13	3	− 3
20	7	− 8	7	− 3	− 5	1
27	4	−14	11	− 1	− 1	1
Oct. 4	8	−16	17	− 2	− 2	− 2
11	16	−11	7	4	0	−13
18	12	−17	5	1	− 6	4
25	8	− 8	4	1	− 3	1
Nov. 1	13	− 6	4	2	− 9	− 6
8	11	− 5	8	− 2	−11	− 2
15	7	−10	12	− 4	− 4	...
22	10	−10	10	−10	− 9	12
29	8	−16	2	6	−15	15
Dec. 6	5	−12	9	− 5	− 8	14
13	13	− 8	15	−10	−10	3
20	11	− 1	12	−14	− 4	− 2
27	− 6	3	19	−24	− 4	12
1921 Jan. 3	− 1	− 1	17	−21	− 5	13
10	− 5	8	12	−15	− 9	12
17	8	− 1	13	− 5	−18	4
24	2	− 8	13	3	−16	5
31	2	−12	11	3	−19	16
Feb. 7	5	− 2	11	− 1	−11	6
14	2	0	10	0	−16	4
21	8	− 1	6	− 5	− 6	...
28	4	− 6	13	− 7	− 2	...
Mar. 7	8	− 1	11	−13	− 3	...
14	12	1	13	−11	− 8	− 8
21	23	− 7	0	0	− 8	−14
28	12	6	4	2	− 1	−24
Apr. 4	13	− 3	6	5	− 9	−10
11	11	0	9	2	−10	−14
18	13	− 3	12	− 7	− 5	−13
25	16	− 7	13	− 6	− 6	−12
May 2	16	−11	13	4	− 8	−11
9	7	−14	14	4	− 7	− 6
16	13	−15	19	− 1	−11	− 7
23	18	−18	12	5	− 9	− 5
30	12	−13	9	3	−10	2



W.T. Comparisons.


1914
1915
1916
1921
(E)
(G)
(P)
+·2
0
−·2
1920
Jan.
Feb.
Mar.
Apr.
May.
June.
July.
Aug.
Sep.
(B)
(E)
(G)
(P)
(U)
(W)

Week commencing	(B).	(E).	(G).	(P).	(U).	(W).
1921 June 6	8	−15	14	6	−11	− 3
13	14	−19	12	4	−11	− 2
20	12	−10	13	− 8	− 8	1
27	12	...	8	8	−25	− 3
July 4	13	−20	14	12	−14	− 2
11	5	−13	15	−11	2	3
18	15	−16	19	−16	− 5	6
25	8	−17	21	−11	− 3	2
Aug. 1	13	−11	23	−11	− 9	− 2
8	13	−10	15	− 4	−14	− 2
15	8	−17	15	6	−10	− 2
22	11	−19	12	12	−17	− 1
29	8	−18	17	13	−15	− 2
Sept. 5	2	−19	18	14	−10	− 2
12	3	−16	17	10	− 8	− 3
19	1	−11	13	12	− 6	− 6
26	1	−14	10	10	− 9	3

Examination of the charts (Plate 4) that express these data can hardly fail to cause considerable surprise, both on account of the sudden changes that take place in them, and, on the other hand, by the way a particular value is adhered to closely for some months and afterwards discarded for another different one. It must be remembered that each of these six observatories takes special pains to determine time. Each observatory has a different record which speaks for itself. Each one was quite unaware of peculiarities in its own record. As it happens, over this period the Paris record is perhaps the most disturbed. In order to verify that it did not unduly modify the mean with which the others are compared, I prepared a second trace in which Paris was eliminated, and the others were compared each with the mean of the remaining five. Certainly the traces become a little smoother, but the characteristic features of each persist. So little does it matter, that I have not thought it necessary to reproduce them. In particular, I would draw attention to the Annapolis (Washington) trace. Ten extremely smooth months are followed by three of rather pronounced oscillation; over the whole of this period the time shows a continuous drift, amounting in all to at least two-tenths of a second fast; then in two months more the whole of this accumulated drift is lost, and a new very consistent run begins and persists for six months, which also shows a definite drift in the original sense.

One serious conclusion I draw from these charts, namely, that they destroy the basis upon which all fine longitude work has hitherto been executed. Thus it has been the custom to take a pair of observers A, B from a place X, and a pair C, D from a place Y. During the first period A, C observe for time at X, while B, D observe at Y. After a

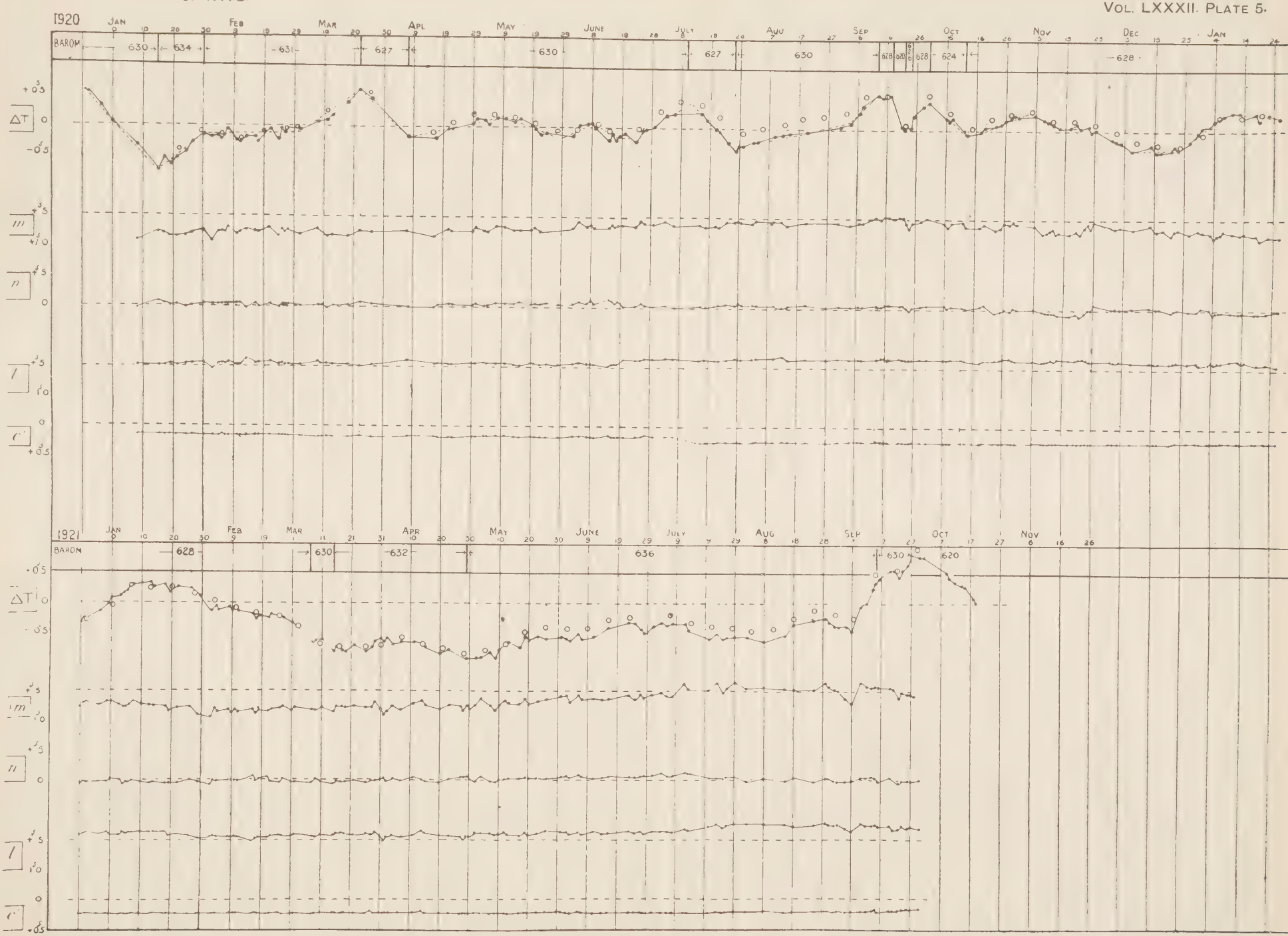
certain interval the observation is reversed, A, C observing at Y, and B, D at X. The aim is, of course, to eliminate personality, but it appears to me that we can have no confidence that such would be the result, whether the two periods are separated by a short interval or a long one. As an illustration we have the recent Washington–Paris determinations, which gave respectively, with observers reversed, $5^h\ 17^m\ 36^s{\cdot}549$ and $36^s{\cdot}758$, a difference for which no explanation could be assigned.

I now turn to the Edinburgh record in particular, to examine what light it throws upon the faults of time determination, whether suspected or unknown.

I have added a smoothed version of the record, to bring out its systematic features. They might be described generally as a pronounced seasonal fluctuation with a minimum about July or August and a maximum about February.

To show how far this is a persistent feature, we have the monthly means for the period 1914–1921 for Greenwich, Paris, and Edinburgh alone. There is some evidence of such a fluctuation in 1919 and 1918; but before that, though possibly present, it is much suppressed. I shall refer further below to this base fluctuation and its possible source.

In Plate 5 I give also unsmoothed the results of the actual time determinations in the form of the ascertained correction required each night by clock Riefler. It is the practice to let this clock run between half a second fast and half a second slow, altering the air pressure from time to time to make the necessary change of rate. The barometric reading is therefore marked as well. The time comparisons brought into the foregoing discussion are "plotted" from Riefler's chart by representing the record by a series of straight lines, as described in my paper in *M.N.*, 1920 November. This crude method is designed, not to secure the closest agreement with the observations, but to leave outstanding and unsuppressed all discordances from the indications of a constant rate for further examination. It will be seen upon examination of these straight lines that upon unfavourable occasions some choice in their position is possible, and I would not say that occasionally a weekly mean so derived may not be off the truth by an amount running up to a limit of, say, $0^s{\cdot}1$. Therefore I do not attach much significance to the small notches in the recorded trace derived from W.T. comparisons, since they could all be removed by an allowable variation of the method of plotting. But the fluctuations which persist when the record is smoothed stand upon a different footing. Small circles on the chart mark the path which the plotted times should occupy in order that the differences *Edinburgh–Mean* may be reduced to zero. It will be seen that the departures are of a character which cannot be explained away by any ordinary theory of accidental error. And, as far as systematic error goes, the system on which the Edinburgh times were found was most carefully kept unvaried. The fluctuation is seasonal. I have at present no conclusion to offer as to its source. The instrumental constants, level, etc., show a similar seasonal fluctuation which would appear in the chart if they fitted badly to the observations. But it is hard to suppose that they fit badly, systematically, for several months together and in the same sense.



Errors of Edinburgh Clock, Riefler 258.

In connection with this subject I may add the preliminary outcome of an inquiry as to how far one can assign the sources of change that are indicated in a chart of clock corrections. Below the clock corrections I have charted the values of m, n, l, c' employed in deriving them. If these were a correct fit to the observations, there should be no sign of correlation between their changes and the changes in the time trace. This is by no means the case. By comparing the detailed features it becomes perfectly clear that the irregular changes in m eliminate themselves very imperfectly from the trace of clock correction; they are, in fact, evidently responsible for most of its erratic changes. On examining further, it is seen that most of these changes in m are due to changes in n; that is, to observations of polar stars. Effects of changes in l, c are also to be traced, but much less frequently. It will, of course, be remembered that the changes shown in the traces of n, l, c' require to be increased by factors approximately 1·5, 1·8, 4·0 respectively before they compare with the changes in ΔT. If this is accepted, it leads to two important conclusions: (1) sudden erratic changes, which appear in the ascertained clock error and are reversed shortly after, have nothing to do with the clock or chronographic system, when these are properly managed, but are introduced into the record by the observations; and (2) such erratic features are due, as a rule, to the adopted constants for correcting the instrumental errors, and not to the time stars. Compare a similar conclusion reached by M. Bigourdan, "Corrections des Signaux Horaires," *Bull. Astron.*, **1** (1921), 260.

The time stars will differ by considerable amounts from one another; it may be taken that ten stars will not agree much within a range of $0^{s}{\cdot}1$, but the mean of their indications will contain little of systematic residual.

It is natural to inquire how far the atmosphere can be held responsible for these irregularities, by means of some anomaly in refraction such as a lateral component. The theory of refraction assumes that the strata of equal density in the atmosphere have a common normal which agrees with the direction of the geographical zenith. This corresponds to the case of an undisturbed gaseous envelope, in which the layers of equal density, pressure, and temperature coincide. In actual circumstances it must often be far from the truth, and the strata of equal density must be oblique to the vertical and in motion. The ray from a star, as it penetrates any stratum, will be refracted towards the zenith of that stratum by an amount $\delta\mu \times \tan$ (zenith distance) where $\delta\mu$ is the increase of refractive index, which is proportional to the increase of density, at the stratum. Now, disregarding at present anything except a deviation of the zenith in the line E – W, and supposing the star to be observed in the meridian, if the ray encounters a succession of strata progressively tilted over more towards the East, from the true geographical zenith Z to the surface inclination Z′, the successive elements of refraction will be the same in amount as in the ordinary theory, but will be directed in turn to points Z_1, Z_2, . . . between Z and Z′. If σ, σ_1, σ_2, . . . σ' be the successive directions of the ray, and if $\sigma\sigma'$ produced meets ZZ′ in ζ, then the effect of the displacement of zenith will be the same as if refraction took

place, not towards Z, but towards ζ, where we need not distinguish between the angular distances σZ, $\sigma\zeta$.

If the image of the star is thus displaced to σ', the lateral amount will be $\sigma\sigma'$ sin $Z\sigma\zeta$

$$= 58''{\cdot}2 \tan \sigma\zeta \sin Z\sigma\zeta = 58''{\cdot}2 \sin Z\zeta \sec \sigma\zeta$$
$$= \rho \sec (\text{zenith distance}) \text{ say, where } \rho = 58''{\cdot}2 \sin Z\zeta.$$

To give a lateral displacement of the order of 1″, the distance of ζ from Z would require to be about one degree.

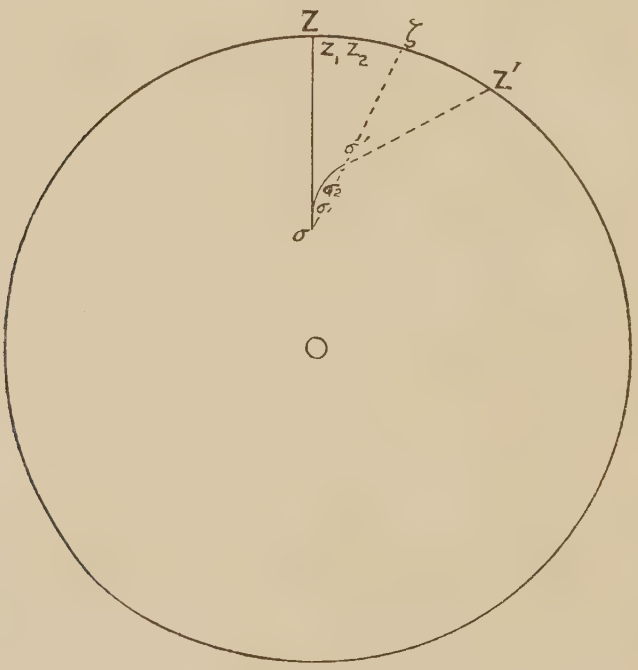


FIG. 1.

Now imagine that collimation and level are determined in the usual way and, let us say, corrected, and that the azimuth correction also happens to be zero, so that the path of the collimation axis of the telescope, which may be taken as a fixed spider thread in the field of view, actually describes the line PZQ, which is the geographical meridian. But in observing a polar at the catalogue time, when it is actually at σ_3 on the meridian, it is seen at σ_3' displaced towards the false zenith, and this is interpreted in the absence of other evidence as a fault in azimuth. Such is the actual process by which the constant n is derived, which is carried into the reduction of observations and used in particular to deduce m, subject to the condition that l, the level error, is zero, as

here, or at least already known. With respect to the false meridian so introduced, the effect upon the transit of the stars will be twofold, first because a false azimuth correction is applied, and second, because the visible transit which is accepted as the observation is that of σ' and not that of σ.

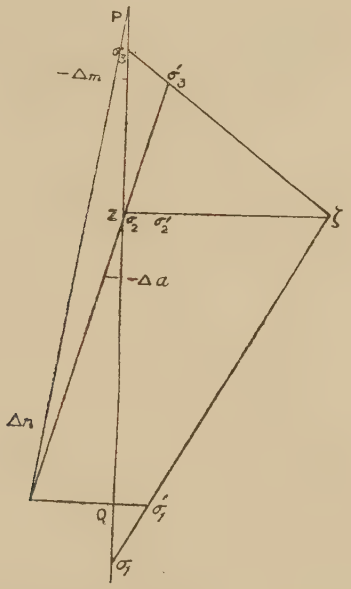


Fig. 2.

If n is taken from a close polar star it is easily seen that

$$-\Delta a = \rho / \sin\phi \cos\phi, \quad \Delta n = \rho / \sin\phi, \quad -\Delta m = \rho / \cos\phi.$$

Again, for an equatoreal star

$$Q\sigma_1' = \rho / \cos\phi.$$

That is to say, if lateral refraction were present, but the solutions were conducted in the usual way, ignoring it, an equatoreal star would seem to transit late by $2\rho \sec\phi$ (in arc) and a zenith star by $\rho \sec\phi$, being half the same amount. Or again, if ΔT_Q is the clock correction derived from equatoreal stars, and ΔT_Z that derived from zenith stars, the clock correction cleared from the effect of lateral refraction would be $2\Delta T_Z - \Delta T_Q$. The slower movement of zenith stars makes this last formula of little service, but by massing a number of observations together it may be seen whether, and by what amount, ΔT_Z stands out generally from ΔT_Q. With this object I have had stars which transit near the zenith regularly observed for clock error along with stars of low declination. The outcome is quite clear. About four times out of five the high stars indicate

the clock slow relative to the equatoreal stars. The means, month by month, are as follows:—

Clock Error—High Stars minus Low Stars.

	J.S.	R.W.W.	E.A.B.	All.	No. of Obs.	Range.
	s	s	s	s		
1920 May . . .	+·03	+·03	+·04	+·03	11	·08
June . . .	·01	·02	·03	·02	14	·09
July . . .	·03	·02	·03	·03	9	·06
August . .	...	·01	·01	·01	6	·02
September . .	·01	+·02	·01	·01	19	·05
October . .	·02	−·01	·02	·01	15	·08
November . .	...	·00	·02	·01	18	·09
December . .	...	+·01	·03	·02	18	·12
1921 January . .	...	·01	·01	·01	17	·07
February . .	...	·01	·01	·01	23	·08
March . .	...	·00	·04	·02	17	·08
April . . .	+·03	·03	·00	·01	11	·07
May . . .	...	·01	·01	+·01	10	·05
June . . .	−·01	+·01	...	−·01	9	·06
July . . .	...	−·01	·00	·00	10	·07
August . .	...	+·01	·02	+·01	7	·05
September . .	·00	·00	·01	·00	15	·05

The observations are surprisingly steady.

The average range, measuring the difference between the greatest positive and lowest negative indication of each month, is $^{s}\cdot07$, and is adhered to pretty closely.

The general mean lag for 229 observations is $^{s}\cdot012$.

If this is interpreted as a lateral displacement of the atmospheric zenith, it gives $\rho = -0''\cdot1$ or ζ permanently west of Z, the geographical zenith, by 6′. The fluctuations from one set of observations to another may reach, say, three times this quantity. These are not negligible amounts; whatever their origin, they would seem sufficient to account for the internal discrepancies of a single set of observations; but they appear to me to exclude most definitely any supposition as to atmospheric effect which would explain the systematic fluctuations, of amplitude about $0^{s}\cdot2$, between the Edinburgh times and the mean of the six observatories.

An accidental displacement East in a polar star, from whatever cause it may arise, of amount $0''\cdot3$, interpreted as a correction to the instrumental constants, would give

$$\Delta a = -0''\cdot6, \quad \Delta n = +^{s}\cdot024, \quad \Delta m = -^{s}\cdot036,$$

and the whole determination of clock correction would be affected approximately by the amount of $-\Delta m$.

That anomalous refractions do occur, anyone can witness who watches the stars with a micrometer on an unsteady night. It may be verified by taking a photographic trail with a fixed telescope. And in fact it has been referred to frequently (cf. "Observations made with the Cookson Floating Zenith Telescope, 1911–18, Royal Observatory, Greenwich").

But there must be great doubt whether a displacement, even so large as 6′, can be entertained between Z and ζ. The barometric gradient in intense storms is about 2·5 mm. of mercury per degree of latitude, which amounts to a gradient of 1′ only in the height of the homogeneous atmosphere. We can hardly suppose a steeper gradient could maintain itself for any length of time or over an extended area. Probably the constant observed difference between high and low stars is no more than a form of personality belonging to the three observers in common, unless indeed it belongs to Boss's P.G.C., to which the places are reduced.

So far as this discussion leads to a definite conclusion it is this—the weakest elements in time determinations are the constants representing the instrumental errors. The clock and chronographic system can be brought to a point of order at which they hardly enter, and it is not difficult to take enough time stars to cancel the accidental error that may attach to any one. If the reduction constants could be made more reliable, a great improvement in smoothness would result. But what steps to take to avoid or to cancel the errors attaching to these constants is a much larger question, upon which I do not touch at present.

But even if another treatment of the instrumental constants reduced time determinations from day to day to complete smoothness, it would still leave unexplained, as far as one can see, the seasonal fluctuation shown by it with respect to the mean of the other observatories. It will be a matter of great interest to see how far the individual features in the traces for different observatories persist or disappear in subsequent years. The fluctuation in the Edinburgh times appears to be seasonal, and I recall that an examination of the temperature coefficients of the Edinburgh Transit Circle (*M.N.*, 1914 December) showed that when the superficial temperature changes were removed, very smooth curves of basic changes emerged which showed seasonal variation in both level and instrumental azimuth, presenting a maximum about February and a minimum about July or August; that is to say, with a lag as though controlled by the temperature a few feet below the surface of the rock. Further, the stellar azimuth derived from observation of polars showed a fluctuation of the same kind with respect to the instrumental azimuth, which was derived from settings upon the N. and S. collimators. These facts are suggestive, but I cannot at present show any reason why they should account for a parallel fluctuation appearing in the time record.

The Use of a Graduated Wedge in Stellar Classification and Parallax Work. By Major William J. S. Lockyer, M.A., Ph.D. (Plate 6.)

The classification of stars according to their spectra is based on the consecutive appearance and disappearance of metallic, ionised, and gaseous lines as the stars increase or decrease their temperature. This system has proved most useful up to the present time, but there remains still a certain amount of uncertainty in the determination of spectral types. A more refined method recently adopted is that in use at the Mount Wilson Observatory, which consists in determining the *numerical* relationships between the intensities of the lines compared, thus providing a more accurate basis for the classification. This method has been recently fully set out,* and need not be described here.

In the measurement of the intensities of the spectral lines, pairs of lines, fairly close together, are selected, the negatives being placed in a stereo-comparator.

Again, the spectroscopic method of determining stellar parallaxes also involves the measurement of the relative intensities of close pairs of lines in the spectrum, for it has been found that the absolute magnitudes of the stars are closely associated with the relative intensities of the spectral lines.†

The present work of the Norman Lockyer Observatory is now devoted to both of these researches by the above-mentioned methods. The Observatory, however, does not possess a stereo-comparator or similar instrument, but has one of the small standard photo-measuring micrometers (Model 13) by Messrs. Adam Hilger. The question arose how to adapt this machine for measuring the relative intensities of the lines to which reference has been made above.

It occurred to the writer that these differences in intensities of close pairs of lines could be determined by means of a graduated neutral-tinted wedge, and, with this idea in mind, a stepped wedge was built up by cutting a homogeneously exposed Kodak film into graduated size strips and placing them one over the other after the principle of an echelon grating.

With this primitive apparatus numerous determinations of the relative intensities of pairs of lines in a spectrum were made. The results were so promising that an order was given to Messrs. Adam Hilger to supply a more refined article.

The wedge was soon received. Its gradated area measured $3{\cdot}8 \times 1{\cdot}95$ cm. It was found, however, that its densest portion was not quite opaque enough to obliterate the strongest lines in some of the spectra measured, so it was determined to procure a further one with a steeper gradient of density.

Both these wedges have now been tested, and they have given such satisfactory results that the researches referred to above have been commenced.

* *Communications to the National Academy of Sciences, Washington*, No. 23, p. 113, 1916, by Walter S. Adams.

† *Ibid.*, No. 24, p. 118, 1916, by Walter S. Adams.

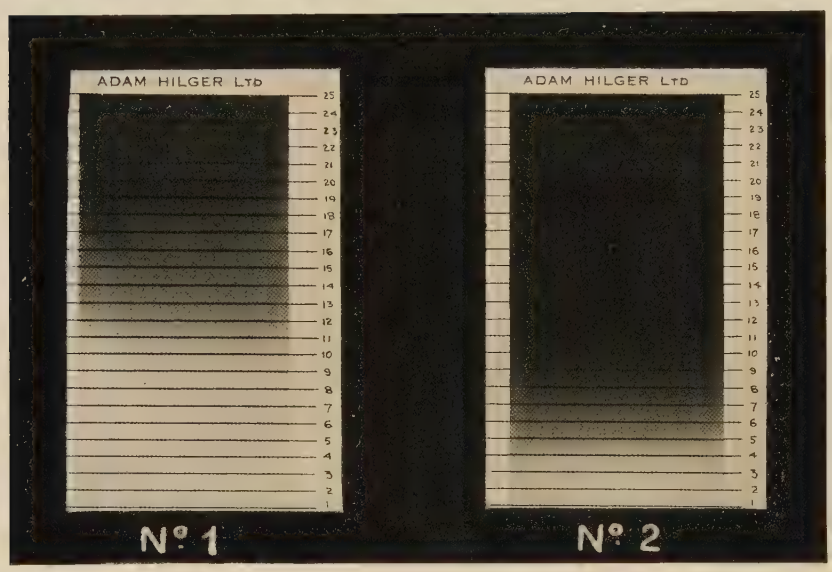


Fig. 1.

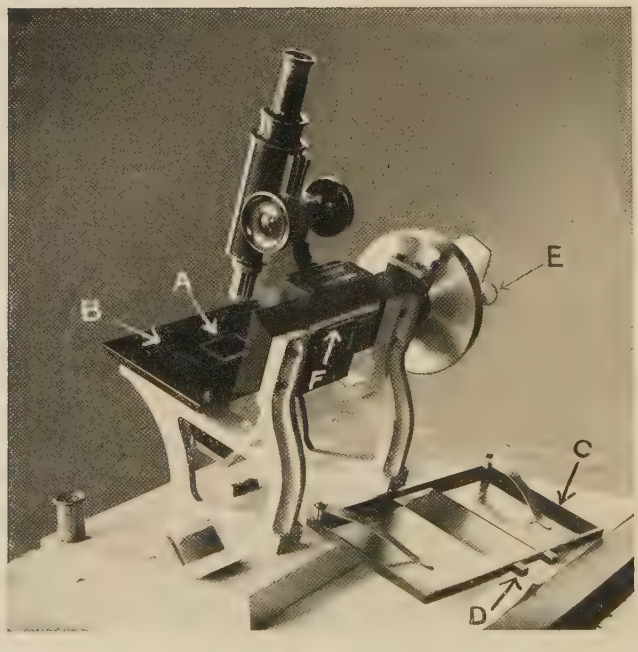


Fig. 2.

The object of this paper is to describe this method now in use, since the apparatus required is simple and comparatively cheap. The method itself has the approval of Professor H. N. Russell, who, after he had examined some of the first results obtained by it, wrote as follows :—

"The method of measurement by means of a photometric wedge which he has devised is new and interesting, and the results show every evidence of great value."

Fig. 1 gives a reproduction of the wedges, and illustrates approximately their different gradations. They are each divided up by lines into twenty-five equal divisions, the width of each measuring about 0·15 cm.

In order to facilitate the measurement of the line intensities, the standard model of the Hilger micrometer had to be slightly modified, but this in no way impaired its use for ordinary measurements. The modifications were made by Messrs. Hilger. The following is a brief account of the whole apparatus as in use at the present time :—

The wedge (A) [see fig. 2] was supported on two small brackets placed at the back of the fixed frame (B), its upper surface being flush with the top of the frame. A carrier (C) was made to hold the negative and slide on the upper surface of the fixed frame (B). This carrier had at the middle of one of its long sides two legs (D) which straddled stiffly, the arm carrying both the microscope and the reflector; it was thus easily detachable. The carrier, microscope, and mirror could all be moved simultaneously by the micrometer handle (E), so that the negative passed over the wedge. Both the negative and the wedge could be viewed at the same time through the microscope.

In order not to interfere with the standard scale of the instrument, an additional scale (F) was fixed corresponding to the divisions on the wedge.

An observation consisted in moving the negative by the micrometer handle (E) until the line in the spectrum was just obliterated and then reading the number on the scale (F), estimating tenths of a division, which corresponded to that portion of the wedge that eclipsed the line.

It was found desirable to work with a more or less constant illumination, to eliminate dull or very bright days: for this purpose the micrometer was placed in a darkened cover on a table, the illumination being supplied by an electric lamp placed just outside a small orifice, the latter being covered with oiled paper.

Test of the Gradation of the Wedges.

In order to test the gradation of the wedges in those portions which fall directly under the optical axis of the microscope, the following procedure was adopted :—

The dense wedge was fixed in the measuring machine and a negative of α Leporis was placed over it in such a way that the Hγ line was in the middle of the first division in the field of view of the microscope. The light wedge was now placed on the negative, so that the first division on it was situated immediately over the Hγ line. The handle of the micrometer was now turned, thus moving the negative, light

wedge, mirror, and microscope over the dense wedge until the line became obliterated. The reading was then taken. The light wedge was now moved along the negative until the middle of the second division was over the Hγ line. The micrometer handle was again turned until the line became obliterated, and the reading taken. This process was continued for all the divisions of the light wedge.

If the wedges were equally graded the advance of one division of the light wedge should correspond to an equal advance on the dense wedge.

Two series of measures were made by two observers, namely, Mr. D. L. Edwards, chief assistant, and Mr. W. B. Rimmer, Research Student

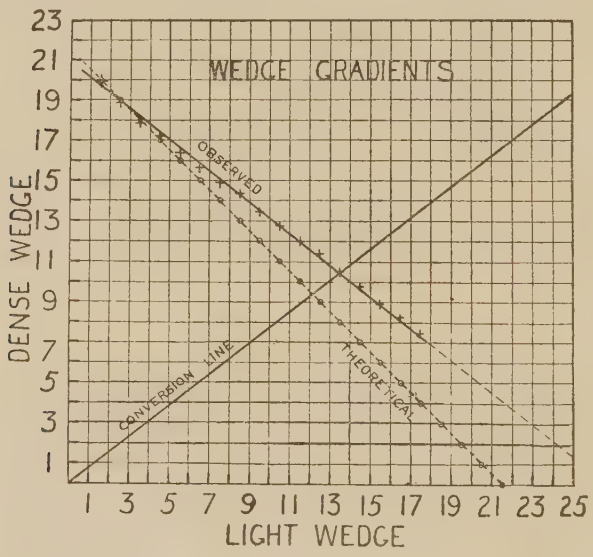


FIG. 3.

appointed by the Scientific and Industrial Research Committee. The means of each were formed, but were so similar that they were combined to form one set of measures. Fig. 3 shows the results obtained.

The broken line, marked theoretical, shows the conditions which should result from similarly gradated wedges. The continuous line with crosses shows the observed readings as the light wedge passed over the dense wedge. The observations are such that a straight line can be drawn passing very closely through the points. Such a condition indicates that the gradients of the wedge are very nearly regular, and not subject to any marked variations of density. The inclination of this line to the theoretical line shows that the gradients are not the same. Thus it will be seen that twenty divisions on the light wedge correspond to 15·5 divisions on the dense wedge, making the relation between the divisions on the light and dense wedges 0·7 to 1 respectively.

Comparison between the Wedges.

There being two wedges of different densities available for use, it was not only interesting to compare measures made with each of them, but it was also important, because, in case any accident should occur to one of them during the process of measuring hundreds of photographs, the work could be carried on with the other.

For this purpose selected lines in the spectrum of γ Orionis, a spectrum with few but well-defined lines, were measured with each wedge by the same two observers.

Three measures were made of each line, the means of these only being given in the following table.

The numbers after the first column represent divisions and estimated tenths of divisions on the scales of the wedges.

Line λ.	Edwards.				Rimmer.			
	Light Wedge.	Dense Wedge.	Diff. L – D.	Residuals from Mean.	Light Wedge.	Dense Wedge.	Diff. L – D.	Residuals from Mean.
1	2	3	4	5	6	7	8	9
Hβ	19·0	9·8	9·2	+0·3	21·8	11·8	10·0	−0·1
4472	18·0	8·0	10·0	−0·5	20·3	10·0	10·3	−0·4
4388	16·0	7·8	9·2	+0·3	17·7	8·0	9·7	+0·2
Hγ	19·3	9·8	9·5	0·0	21·5	10·9	10·6	−0·7
4144	17·8	8·0	9·8	−0·3	19·3	10·4	8·9	+1·0
4121	14·3	5·3	9·0	+0·5	17·3	7·6	9·7	+0·2
Hδ	21·0	10·8	10·2	−0·7	22·5	12·2	10·3	−0·4
4026	17·3	7·0	10·3	−0·8	18·7	9·8	8·9	+1·0
4009	14·6	5·3	8·7	+0·8	17·4	7·3	10·1	−0·2
Hε	19·0	9·8	9·2	+0·3	22·0	11·5	10·5	−0·6
Means	17·6	8·2	9·5		19·8	9·9	9·9	

Column 1 in the above table indicates the wave-lengths or names of the lines measured. Columns 2 and 6 give the measures of the various lines with the light wedge by the two observers. It will be seen that Mr. Rimmer's values are almost constantly larger than those of Mr. Edwards' by a mean amount of 2·2 wedge divisions. Similar remarks apply to the measures of the dense wedge in columns 3 and 7, only Mr. Rimmer's measures are here 1·7 divisions larger than those of Mr. Edwards'.

In spite, however, of this personal equation, the individual differences between the readings of the light and dense wedges, as shown in columns 4 and 8, by both observers, are in good agreement, the mean values in the two cases being 9·5 and 9·9 divisions for Mr. Edward and Mr. Rimmer respectively. The residuals, as shown in columns 5 and 9, are one division, or less than one division, of the wedges. The

result of this inquiry shows that the light-wedge readings are in the mean 9·7 divisions more than those of the dense wedge.

The probable error of a single measure of the difference between the wedges is ±0·37 and ±0·41 divisions, and that for the mean difference of all the measures is ±0·12 and ±0·13 divisions for Mr. Edwards and Mr. Rimmer respectively.

Measures of the Differences of Intensities between Close Pairs of Spectral Lines.

In order to form an idea of the accuracy of this method in determining the differences in intensities between close pairs of lines in a stellar spectrum, a series of measures of six selected pairs of lines in four stellar spectra were made. The negatives used were those obtained with the Frank McClean 12-inch objective-prism telescope, and belong to the series of negatives now being measured for parallax determinations.

The measures were made by the same two observers. The following table shows the stars used, the selected pairs of lines, and the measures by the two observers, each measure being the mean of five readings:—

λ of Pairs.	α Cassiopeiæ.			β Herculis.			γ Cephei.			τ Ceti.		
	Edwards.	Rimmer.	E−R.	Edwards.	Rimmer.	E−R.	Edwards.	Rimmer.	E−R.	Edwards.	Rimmer.	E−R.
$\frac{4227}{4216}$	1·5	1·3	+0·2	1·5	1·7	−0·2	2·4	2·7	−0·3	4·0	4·3	−0·3
$\frac{4216}{4250}$	3·9	4·0	−0·1	3·5	3·5	0·0	2·2	2·5	−0·3	0·4	0·1	+0·3
$\frac{4227}{H\gamma}$	3·0	3·3	−0·3	2·5	2·6	−0·1	4·7	5·0	−0·3	1·7	1·7	0·0
$\frac{4326}{4272}$	1·7	1·5	+0·2	2·5	3·0	−0·5	1·0	0·6	+0·4	1·3	1·7	−0·4
$\frac{4326}{H\gamma}$	1·8	1·5	+0·3	1·5	1·9	−0·4	2·5	2·6	−0·1	1·0	1·3	−0·3
$\frac{H\gamma}{4352}$	1·3	1·4	−0·1	1·0	1·0	0·0	0·5	0·2	+0·3	1·6	2·0	−0·4
Means	2·2	2·1	...	2·1	2·3	...	2·2	2·3	...	1·6	1·8	...

By examining the differences between the measures of the two observers, it will be seen that they are all equal to, or less than, 0·4 of a division, with one exception. This exception occurs in the measurement of the fourth pair, the difference being −0·5.

A reference to this pair on a large dispersion spectrum shows that while λ4326 is a good line to measure, λ4272 is not satisfactory, as it has on its red side two lines close to it due to Fe and Cr. This composite nature, more or less blended on a negative of smaller dispersion, most probably accounts for the larger values recorded, and indicates that such a pair of lines should not have been used. Neglecting these

values and remembering that tenths of a division are only estimated on the wedge scale, the resulting measures are very good.

Interconversion of the Wedge Divisions.

By drawing a line, as shown in fig. 3, between the zero of the wedge divisions and the point where 20 on the light wedge meets division 15·5 on the dense wedge, any point on this line gives the relation between a division on the light wedge and the corresponding division on the dense wedge. Thus the differences in intensities between two lines measured on one wedge can be rapidly converted into those on the other.

To test this, the measured pairs of lines in the four stars referred to above were now measured similarly with the light wedge. These latter measures were then converted to the dense-wedge divisions by means of the above-described scale. The means of the same two observers in each case were compared, and the results are shown in the following table :—

	α Cassiopeiæ.			β Herculis.			γ Cephei.			τ Ceti.		
	Observed.	Reduced.	O – R.	Observed.	Reduced.	O – R.	Observed.	Reduced.	O – R.	Observed.	Reduced.	O – R.
4227/4216	1·4	1·7	−0·3	1·6	1·6	0·0	2·55	2·5	+0·05	4·15	4·2	−0·05
4216/4250	3·95	3·6	+0·35	3·5	3·5	0·0	2·35	2·2	+0·15	0·25	0·15	+0·1
4227/Hγ	3·15	2·6	+0·55	2·55	2·4	+0·15	4·85	4·4	+0·45	1·7	1·8	−0·1
4326/4272	1·6	1·3	+0·3	2·75	2·7	+0·05	1·3	0·9	+0·4	1·5	1·8	−0·3
4326/Hγ	1·65	1·3	+0·35	1·7	1·5	+0·2	2·55	2·4	+0·15	1·15	1·2	−0·05
Hγ/4352	1·35	1·5	−0·15	1·0	0·8	+0·2	0·35	0·3	+0·05	1·8	1·6	+0·2

The residuals shown in the columns under the heading O – R are very satisfactory, the largest being only 0·55. The mean value of all the residuals, regardless of sign, is 0·19 divisions, a quantity well within the error of measurement. The conclusion to be derived is that, should the necessity arise to convert one scale into another, the conversion diagram given above presents a very accurate method.

Effect of Errors in Measurement on the final Results.

In the measurement of the differences in intensities between close pairs of lines it is found that ±0·4 of a division is about the largest error that will enter into the final determinations of stellar classification and parallax from a *single* measure.

At the present time it is not possible to say how the error will affect the cases of stars of every spectrum type, because curves for each type are not available. It is possible, however, to state the case in

stars of F and K types, for these have now been determined at the Norman Lockyer Observatory.

In the case of F-type stars an error of $\pm 0{\cdot}5$ division in the wedge readings will produce an error of $\pm 0{\cdot}5$ in the absolute magnitude, and of about 25 per cent. in the parallax. For K-type stars the error in absolute magnitude would be $\pm 1{\cdot}0$, and in parallax about 50 per cent.

The determinations of the differences in intensities of the pairs of lines is, however, the *mean* of at least three sets of measures, and, further, three distinct pairs of lines are in general used for the determination of the absolute magnitude of each star. The probable errors in the final results are therefore very considerably reduced from those stated above.

Correction to the Catalogue of William Herschel Papers, *Monthly Notices*, **78**, 548:—

Part XI. of the "Journal," supposed to be lost, has recently been found at Observatory House, Slough, and has been added to the collection. It contains observations from May 19, 1787, to March 7, 1794, chiefly of planets and satellites.